Math 250 Exam #4 Key (Spring 2014)

1 We have

$$\begin{aligned}\mathcal{L}[f](s) &= \int_0^\infty e^{-st} f(t)\,dt = \int_0^3 e^{(2-s)t}\,dt + \int_3^\infty e^{-st}\,dt \\ &= \frac{1}{2-s}\left[e^{(2-s)t}\right]_0^3 + \lim_{b\to\infty}\int_3^b e^{-st}\,dt = \frac{e^{3(2-s)}-1}{2-s} + \lim_{b\to\infty} -\frac{1}{s}\left[e^{-st}\right]_3^b \\ &= \frac{e^{6-3s}-1}{2-s} - \frac{1}{s}\lim_{b\to\infty}\left(e^{-sb}-e^{-3s}\right) = \frac{e^{6-3s}-1}{2-s} - \frac{1}{s}\left(0-e^{-3s}\right)\end{aligned}$$

if $s > 0$ and $s \neq 2$. Thus

$$\mathcal{L}[f](s) = \frac{e^{6-3s}-1}{2-s} + \frac{e^{-3s}}{s}, \quad s \in (0,2)\cup(2,\infty).$$

2 Using the table provided,

$$\mathcal{L}\left[t^8 e^{-6t}\right](s) = \frac{8!}{(s+6)^9} = \frac{40,320}{(s+6)^9}.$$

3 Using the table provided,

$$\begin{aligned}\mathcal{L}\left[t^5 - 7e^{-3t}\sin 4t\right](s) &= \mathcal{L}\left[t^5\right](s) - 7\mathcal{L}\left[e^{-3t}\sin 4t\right](s) \\ &= \frac{5!}{(s-0)^{5+1}} - 7\cdot\frac{4}{(s+3)^2+4^2} \\ &= \frac{120}{s^6} - \frac{28}{(s+3)^2+16}\end{aligned}$$

4 Use a trigonometric identity for this, along with the identity $\sin(-u) = -\sin(u)$ and the transform table:

$$\begin{aligned}\mathcal{L}[\sin 3t\cos 7t](s) &= \mathcal{L}\left[\tfrac{1}{2}\sin(10t) + \tfrac{1}{2}\sin(-4t)\right](s) = \tfrac{1}{2}\mathcal{L}[\sin(10t)](s) - \tfrac{1}{2}\mathcal{L}[\sin(4t)](s) \\ &= \frac{1}{2}\cdot\frac{10}{s^2+10^2} - \frac{1}{2}\cdot\frac{4}{s^2+4^2} = \frac{5}{s^2+100} - \frac{2}{s^2+16}.\end{aligned}$$

5 Partial fraction decomposition is necessary: we have

$$\frac{s-4}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2},$$

which works out to give $A = 1$ and $B = -7$, and so

$$\mathcal{L}^{-1}\left[\frac{s-4}{(s+3)^2}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s+3}\right](t) - 7\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2}\right](t) = e^{-3t} - 7te^{-3t} = (1-7t)e^{-3t}.$$

6 We have

$$\mathcal{L}[y''](s) - 2\mathcal{L}[y'](s) - \mathcal{L}[y](s) = \mathcal{L}\big[e^{2t}\big](s) - \mathcal{L}\big[e^{t}\big](s).$$

Letting $Y(s) = \mathcal{L}[y(t)](s)$ and using relevant properties gives

$$[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] - Y(s) = \frac{1}{s-2} - \frac{1}{s-1}.$$

Enter the initial conditions, and let Y stand for $Y(s)$, to get

$$s^2Y - s - 3 - 2sY + 2 - Y = \frac{1}{(s-1)(s-2)}.$$

and then

$$(s^2 - 2s - 1)Y = \frac{1}{(s-1)(s-2)} + s + 1,$$

and finally

$$Y(s) = \frac{1}{(s-1)(s-2)(s^2-2s-1)} + \frac{s+1}{s^2-2s-1}.$$

7 Let $w(t) = y(t+2)$, so $w'(t) = y'(t+2)$ and $w''(t) = y''(t+2)$, and also $w(0) = y(2) = 3$ and $w'(0) = y'(2) = 0$. Substituting $t+2$ for t in the original ODE, we obtain the IVP

$$w''(t) - w(t) = t, \quad w(0) = 3, \quad w'(0) = 0.$$

Let $W = \mathcal{L}[w](s)$, so

$$\mathcal{L}[w''](s) - \mathcal{L}[w](s) = \mathcal{L}[t](s) \;\;\Rightarrow\;\; [s^2W - sw(0) - w'(0)] - W = \frac{1}{s^2} \;\;\Rightarrow\;\; s^2W - 3s - W = \frac{1}{s^2},$$

and hence

$$W(s) = \frac{1}{s^2(s^2-1)} + \frac{3s}{s^2-1} = \frac{3s^3+1}{s^2(s-1)(s+1)} = \frac{2}{s-1} + \frac{1}{s+1} - \frac{1}{s^2}.$$

From this we obtain

$$w(t) = \mathcal{L}^{-1}[W](t) = 2\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) = 2e^t + e^{-t} - t.$$

Since $y(t) = w(t-2)$, we finally arrive at the solution:

$$y(t) = 2e^{t-2} - e^{2-t} - t + 2.$$