Math 250 Exam #3 Key (Spring 2014)

1a We have $y'' + y' + 4y = e^t + e^{-t}$.

$$\text{Auxiliary equation: } r^2 + r + 4 = 0; \quad \text{roots: } -\frac{1}{2} \pm \frac{\sqrt{15}}{2} i.$$

Start with equation $y'' + y' + 4y = e^t$, with nonhomogeneity $f(t) = e^t$. Since $\alpha = 1$ is not a root of the auxiliary equation, a particular solution has form $y_1(t) = Ae^t$. Substituting into ODE:

$$Ae^t + (Ae^t)' + 4(Ae^t)'' = e^t \quad \Rightarrow \quad 6Ae^t = e^t \quad \Rightarrow \quad A = \frac{1}{6},$$

so $y_1(t) = \frac{1}{6}e^t$.

Next we have $y'' + y' + 4y = e^{-t}$, with nonhomogeneity $f(t) = e^{-t}$. Since $\alpha = -1$ is not a root of the auxiliary equation, a particular solution has form $y_2(t) = Ae^{-t}$. Substituting into ODE:

$$Ae^{-t} + (Ae^{-t})' + 4(Ae^{-t})'' = e^{-t} \quad \Rightarrow \quad 4Ae^{-t} = e^{-t} \quad \Rightarrow \quad A = \frac{1}{4},$$

so $y_2(t) = \frac{1}{4}e^{-t}$.

By Superposition Principle a particular solution to the original equation is

$$y_p(t) = \frac{1}{6}e^t + \frac{1}{4}e^{-t}.$$

The general solution is thus

$$y(t) = \frac{1}{6}e^t + \frac{1}{4}e^{-t} + e^{-t/2}\left(c_1 \cos \frac{\sqrt{15}}{2}t + c_2 \sin \frac{\sqrt{15}}{2}t\right).$$

1b The auxiliary equation $r^2 + 2r + 5 = 0$ has roots $-1 \pm 2i$. The nonhomogeneity has the factor $e^{\alpha t} \cos \beta t$ with $\alpha - 1$ and $\beta = 2$, and since $\alpha + \beta i = -1 + 2i$ is a root of the auxiliary equation, we take $s = 1$ in the Method of Undetermined Coefficients. Thus a particular solution has form

$$y_p(t) = Ate^{-t} \cos 2t + Bte^{-t} \sin 2t.$$

Now $y_p''(t) + 2y_p'(t) + 5y_p(t) = 4e^{-t} \cos 2t$ becomes

$$(-4A \sin 2t + 4B \cos 2t)e^{-t} = (4 \cos 2t)e^{-t},$$

which shows that $A = 0$ and $B = 1$, and therefore

$$y_p(t) = te^{-t} \sin 2t.$$

The general solution is thus

$$y(t) = te^{-t} \sin 2t + e^{-t}(c_1 \cos 2t + c_2 \sin 2t). \tag{1}$$

Next, substituting the initial condition $y(0) = 1$ into (1) yields $c_1 = 1$. Substituting the initial condition $y'(0) = 0$ into the derivative of (1) yields $c_2 = \frac{1}{2}$. We now obtain the solution to the IVP:

$$y(t) = te^{-t} \sin 2t + e^{-t}\big(\cos 2t + \tfrac{1}{2} \sin 2t\big)$$

2 The auxiliary equation $r^2-2r+1=0$ has double root 1, so the corresponding homogeneous equation $y''-2y'+1=0$ has linearly independent solutions $y_1(t)=e^t$ and $y_2(t)=te^t$, and thus

$$y_1'(t)=e^t \quad \text{and} \quad y_2'(t)=(1+t)e^t.$$

Now, since $a_2=1$ and $f(t)=(1+t^2)^{-1}e^t$, we have

$$v_1(t)=\int \frac{-te^t\cdot(1+t^2)^{-1}e^t}{e^{2t}}dt=-\int \frac{t}{1+t^2}dt=-\frac{1}{2}\ln(1+t^2),$$

and

$$v_2(t)=\int \frac{e^t\cdot(1+t^2)^{-1}e^t}{e^{2t}}dt=\int \frac{1}{1+t^2}dt=\tan^{-1}(t).$$

Hence a particular solution to the ODE is

$$y_p(t)=v_1(t)y_1(t)+v_2(t)y_2(t)=-\tfrac{1}{2}e^t\ln(1+t^2)+te^t\tan^{-1}(t).$$

General solution is therefore

$$y(t)=\left[-\tfrac{1}{2}\ln(1+t^2)+t\tan^{-1}(t)+c_1t+c_2\right]e^t.$$

3a From the IVP

$$\tfrac{1}{8}y''+2y'+16y=0, \quad y(0)=-\tfrac{3}{4}, \quad y'(0)=-2$$

comes the equation of motion:

$$y(t)=\left(-\tfrac{3}{4}\cos 8t-\sin 8t\right)e^{-8t}. \tag{2}$$

3b Find the smallest $t>0$ such that $y'(t)=0$. Differentiating (2), we get

$$(6\sin 8t-8\cos 8t)e^{-8t}-8\left(-\tfrac{3}{4}\cos 8t-\sin 8t\right)e^{-8t}=0,$$

whence

$$7\sin 8t-\cos 8t=0. \tag{3}$$

Any value for t that gives $\cos 8t=0$ will give $\sin 8t=\pm 1$ and hence will not be a solution to (3), so we can safely divide (3) by $\cos 8t$ to obtain $\tan 8t=\frac{1}{7}$, and thus the smallest positive solution is

$$t=\tfrac{1}{8}\tan^{-1}\left(\tfrac{1}{7}\right)\approx 0.01774.$$

That is, the object is maximally displaced to the left at time $t=0.01774$ s, when the displacement is

$$y(0.01774)=-0.767 \text{ m}$$

by equation (2).

3c To find the quasiperiod, just double the time elapsed between two successive instances when the object passes through the equilibrium position $y=0$. Setting $y(t)=0$, by (2) we have

$$-\tfrac{3}{4}\cos 8t-\sin 8t=0 \;\Rightarrow\; 3+4\tan 8t=0 \;\Rightarrow\; \tan 8t=-\tfrac{3}{4}.$$

The function $\tan 8t$ is periodic with period $\pi/8$, and so the quasiperiod is $\pi/4$ and the quasi-frequency is $4/\pi$.