Math 250 Exam #2 Key (Spring 2014)

1 We have

$$\frac{N_x(x,y) - M_y(x,y)}{M(x,y)} = \frac{2y-1}{y} = 2 - \frac{1}{y}$$

is a function of only y, so

$$\mu(y) = \exp\left(\int (2 - 1/y)\,dy\right) = e^{2y - \ln y} = y^{-1}e^{2y}.$$

Multiplying the ODE by $y^{-1}e^{2y}$ yields the exact equation

$$e^{2y} + (2xe^{2y} - y^{-1})y' = 0.$$

There exists a function F such that

$$F_x(x,y) = e^{2y} \quad \text{and} \quad F_y(x,y) = 2xe^{2y} - y^{-1}.$$

From the former equation comes

$$F(x,y) = xe^{2y} + g(y),$$

so the latter equation implies

$$2xe^{2y} + g'(y) = 2xe^{2y} - y^{-1} \;\Rightarrow\; g'(y) = -y^{-1} \;\Rightarrow\; g(y) = -\ln|y| + c_1,$$

c_1 arbitrary. The general solution to the ODE is $F(x,y) = c_2$, where c_2 is arbitrary. That is,

$$xe^{2y} - \ln|y| = c,$$

where c is the arbitrary constant deriving from $c_2 - c_1$.

2 Multiply the ODE by $x^m y^n$:

$$(x^{m+3}y^{n+2} - 2x^m y^{n+3}) + (x^{m+4}y^{n+1})y' = 0.$$

For exactness we need $M_y = N_x$, or

$$(n+2)x^{m+3}y^{n+1} - 2(n+3)x^m y^{n+2} = (m+4)x^{m+3}y^{n+1}.$$

Matching coefficients of like terms, we find that we must have $n+2 = m+4$ and $-2(n+3) = 0$, which solves to give $m = -5$ and $n = -3$. Thus an integrating factor is $\mu(x,y) = x^{-5}y^{-3}$, and the ODE becomes

$$(x^{-2}y^{-1} - 2x^{-5}) + (x^{-1}y^{-2})y' = 0,$$

which is exact. We now find a function F such that $F_x(x,y) = x^{-2}y^{-1} - 2x^{-5}$ and $F_y(x,y) = x^{-1}y^{-2}$. From the former equation we obtain

$$F(x,y) = \int \left(x^{-2}y^{-1} - 2x^{-5}\right) dx = -x^{-1}y^{-1} + \tfrac{1}{2}x^{-4} + g(y),$$

and from the latter equation comes

$$x^{-1}y^{-2} = F_y(x,y) = x^{-1}y^{-2} + g'(y),$$

or $g'(y) = 0$. Thus $g(y) = c_1$ for some constant c_1, and we have

$$F(x,y) = -x^{-1}y^{-1} + \tfrac{1}{2}x^{-4} + c_1.$$

The implicit solution to the ODE is therefore $F(x,y) = c_2$ for arbitrary constant c_2, which we may write simply as

$$\tfrac{1}{2}x^{-4} - x^{-1}y^{-1} = c$$

by merging constant terms. Another solution is $y \equiv 0$.

3 Rewrite ODE as

$$y' = -\frac{x^2+y^2}{2xy} = -\frac{1+(y/x)^2}{2(y/x)}.$$

Let $v = y/x$, so $y = xv$ and we get $y' = xv' + v$. ODE then becomes

$$xv' + v = -\frac{1+v^2}{2v},$$

which is separable and so leads to

$$\int \frac{1}{-(1+v^2)/(2v) - v}\,dv = \int \frac{1}{x}\,dx.$$

A little algebra then gives

$$-\int \frac{2v}{3v^2+1}\,dv = \int \frac{1}{x}\,dx.$$

Making the substitution $w = 3v^2+1$, the integral on the left-hand side transforms to give

$$-\int \frac{1/3}{w}\,dw = \int \frac{1}{x}\,dx,$$

and hence

$$-\frac{1}{3}\ln|w| = \ln|x| + c_1$$

for any constant c_1. From $w = 3v^2+1 = 3y^2/x^2+1$ comes the general implicit solution

$$\ln\left(\frac{3y^2}{x^2}+1\right) = -3\ln|x| + c_1,$$

which can be rearranged to give

$$3\ln|x| + \ln\left(\frac{3y^2}{x^2}+1\right) = \ln|x^3| + \ln\left(\frac{3y^2}{x^2}+1\right) = \ln\left(3|x|y^2 + |x|x^2\right) = c_1.$$

To get the general solution in a nicer form, we can exponentiate the last equality to get

$$|x|(3y^2+x^2) = c_2,$$

where $c_2 = \exp(c_1) > 0$ is arbitrary. From this comes $x(3y^2+x^2) = \pm c_2$, and so we may as well replace $\pm c_2$ with the arbitrary constant $c \neq 0$ and write

$$x^3 + 3xy^2 = c.$$

4 The equation is Bernoulli with $n = 2$, $P(x) = -2/x$, and $Q(x) = -x^2$. Letting $v = y^{1-n} = y^{-1}$, we obtain the linear equation

$$v' + \frac{2}{x}v = x^2.$$

Multiplying by the integrating factor

$$\mu(x) = e^{\int 2/x\,dx} = e^{\ln(x^2)} = x^2$$

yields

$$x^2v' + 2xv = x^4 \quad\Rightarrow\quad (x^2v)' = x^4 \quad\Rightarrow\quad x^2v = \tfrac{1}{5}x^5 + c \quad\Rightarrow\quad v = \tfrac{1}{5}x^3 + cx^{-2},$$

and finally

$$y^{-1} = \tfrac{1}{5}x^3 + cx^{-2} \qquad \text{or} \qquad y = \frac{5x^2}{x^5 + c}.$$

Also $y \equiv 0$ is a solution.

5 Let $x(t)$ be the mass of salt, in kilograms, in the tank at time t, so that $x(0) = 30$. The volume of solution in the tank is $V(t) = 200 + 2t$. The full derivation of $x'(t)$, which is the rate of change of the amount of salt in the tank at time t, is as follows:

$$\begin{aligned} x'(t) &= (\text{rate salt enters Tank 1}) - (\text{rate salt leaves Tank 1}) \\ &= \left(\frac{0.3\text{ kg}}{1\text{ L}}\right)\left(\frac{4\text{ L}}{1\text{ min}}\right) - \left(\frac{x(t)\text{ kg}}{V(t)\text{ L}}\right)\left(\frac{2\text{ L}}{1\text{ min}}\right) \\ &= 1.2 - \frac{2x(t)}{200 + 2t}. \end{aligned}$$

Thus we have a linear first-order ODE:

$$x' + \frac{x}{t + 100} = \frac{6}{5}.$$

To solve this equation, we multiply by the integrating factor

$$\mu(t) = \exp\left(\int \frac{1}{t + 100}\,dt\right) = e^{\ln(t+100)} = t + 100$$

to obtain

$$(t + 100)x' + x = \frac{6}{5}(t + 100),$$

which becomes

$$[(t + 100)x]' = \frac{6}{5}(t + 100)$$

and thus

$$(t + 100)x = \frac{6}{5}\int (t + 100)\,dt = \frac{3}{5}t^2 + 120t + c.$$

From this we get a general explicit solution to the ODE,

$$x(t) = \frac{3t^2 + 600t + c}{5t + 500}.$$

To determine c we use the initial condition $x(0) = 30$, giving $c/500 = 30$, and thus $c = 15,000$. So, the amount of salt in the tank at time t is given by

$$x(t) = \frac{3t^2 + 600t + 15,000}{5t + 500}.$$

The tank is full when $t = 150$ minutes. At that time the concentration of salt is:
$$\frac{x(150)}{V(150)} = \frac{1}{500}\left(\frac{3(150)^2 + 600(150) + 15,000}{5(150) + 500}\right) = \frac{138}{500} = 0.276 \text{ kg/L}.$$

6 Auxiliary equation is $2r^2 + 7r - 15 = 0$, which has solutions $r = \frac{3}{2}, -5$. Hence
$$y(t) = c_1 e^{3t/2} + c_2 e^{-5t}.$$
Now,
$$y'(t) = \tfrac{3}{2} c_1 e^{3t/2} - 5c_2 e^{-5t},$$
and so the initial conditions give
$$c_1 + c_2 = -2 \quad \text{and} \quad \tfrac{3}{2} c_1 - 5c_2 = 4.$$
Solving the system gives $c_1 = -\frac{12}{13}$ and $c_2 = -\frac{14}{13}$, and therefore
$$y(t) = -\tfrac{12}{13} e^{3t/2} - \tfrac{14}{13} e^{-5t}.$$

7 Auxiliary equation is $9r^2 - 12r + 4 = 0$, which has double root $r = 2/3$. General solution is therefore
$$y(t) = c_1 e^{2t/3} + c_2 t e^{2t/3}.$$

8 Auxiliary equation is $12r^3 - 28r^2 - 3r + 7 = 0$, which factors by grouping:
$$(3r - 7)(2r - 1)(2r + 1) = 0.$$
Roots are $r = 7/3, 1/2, -1/2$. General solution:
$$y(t) = c_1 e^{-t/2} + c_2 e^{t/2} + c_3 e^{7t/3}.$$

9 Auxiliary equation is $r^2 + 9 = 0$, which has roots $r = \pm 3i$. Thus we have
$$y(t) = c_1 \cos 3t + c_2 \sin 3t$$
With the initial conditions $y(0) = 1$ and $y'(0) = 1$ we find that $c_1 = 1$ and $c_2 = 1/3$, and so
$$y(t) = \cos 3t + \tfrac{1}{3} \sin 3t.$$