Math 250 Exam #1 Key (Spring 2014)

1 Substitute $\varphi(x) = x^m$ for y to obtain

$$\begin{aligned} 3x^2 \cdot m(m-1)x^{m-2} + 11x \cdot mx^{m-1} - 3x^m &= 0 \\ (3m^2 - 3m)x^m + 11mx^m - 3x^m &= 0 \\ (3m^2 + 8m - 3)x^m &= 0 \end{aligned}$$

To satisfy the equation for all x in some interval $I \subseteq \mathbb{R}$, it will be necessary to have

$$3m^2 + 8m - 3 = 0.$$

Solving this equation for m, we have

$$(3m-1)(m+3) = 0$$

and thus $m = 1/3, -3$. This shows that $\varphi_1(x) = \sqrt[3]{x}$ and $\varphi_2(x) = x^{-3}$ are solutions to the ODE (each valid on $(-\infty, 0) \cup (0, \infty)$, incidentally).

2 We have

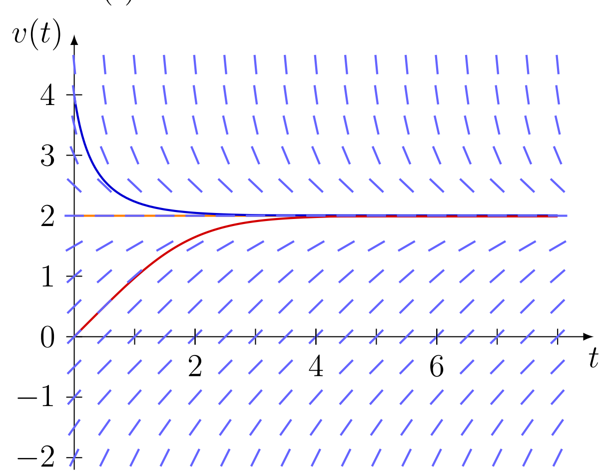
$$f(x,y) = 4x^2 + \sqrt[3]{2-y},$$

which is continuous throughout $\mathbb{R}^2$. However

$$f_y(x,y) = \frac{1}{3}(2-y)^{-2/3} = \frac{1}{3\sqrt[3]{(2-y)^2}}$$

is not continuous on the line $y = 2$ since f_y is not defined there. The initial point $(-1, 2)$ lies on this line, and so the Existence-Uniqueness Theorem does not imply that the initial value problem has a unique solution.

3 The solution curves corresponding to the initial conditions $v(0) = 0$, $v(0) = 2$, and $v(0) = 4$ are below. It can be seen that $v(t) \to 2$ as $t \to \infty$.



4 We are given $(x_0, y_0) = (1, 0)$ and $h = 0.2$.

n	0	1	2	3	4
x_n	1.0	1.2	1.4	1.6	1.8
y_n	0	0	0	0	0

5 Rewriting the equation as

$$y' = -y^2 e^{\cos x} \sin x$$

shows it to be separable. We obtain

$$-\int \frac{1}{y^2} dy = \int e^{\cos x} \sin x \, dx.$$

Let $u = \cos x$ in the integral at right, so

$$\frac{1}{y} = -\int e^u du = -e^u + c = -e^{\cos x} + c$$

for arbitrary constant c. That is,

$$y = \frac{1}{c - e^{\cos x}}.$$

6 Writing the equation as $y' = 2\cos x\sqrt{y+1}$, we see the equation is separable. We get

$$\int \frac{1}{\sqrt{y+1}} \, dy = \int 2\cos x \, dx.$$

This integrates easily to give

$$2\sqrt{y+1} = 2\sin x + c.$$

Now, $y(\pi) = 0$ implies that

$$2\sqrt{0+1} = 2\sin \pi + c,$$

or $c = 2$. The (implicit) solution to the IVP is thus

$$2\sqrt{y+1} = 2\sin x + 2,$$

or $y = (\sin x + 1)^2 - 1$.

7 The equation may be written as

$$y' + \frac{3}{x} y = \frac{\sin x}{x^2} - 3x,$$

which is the standard form for a 1st-order linear ODE. An integrating factor is

$$\mu(x) = e^{\int 3/x \, dx} = e^{3 \ln x} = x^3.$$

Multiplying the ODE by x^3 gives

$$x^3 y' + 3x^2 y = x \sin x - 3x^4,$$

which becomes $(x^3y)' = x\sin x - 3x^4$ and thus

$$x^3y = \int x\sin x\,dx - \frac{3}{5}x^5 + c.$$

Integration by parts gives

$$\int x\sin x\,dx = \sin x - x\cos x,$$

so that $x^3y = \sin x - x\cos x - \frac{3}{5}x^5 + c$ and therefore

$$y(x) = \frac{1}{x^3}\left(\sin x - x\cos x - \frac{3}{5}x^5 + c\right).$$

is the general solution.

8 The equation may be written as

$$x' + \frac{3}{t}x = t^2\ln t + \frac{1}{t^2},$$

which is the standard form for a 1st-order linear ODE. An integrating factor is

$$\mu(x) = e^{\int 3/t\,dt} = t^3.$$

Multiplying the ODE by t^3 gives $t^3x' + 3t^2x = t^5\ln t + t$, which becomes $(t^3x)' = t^5\ln t + t$ and thus

$$t^3x = \int t^5\ln t\,dt + \frac{1}{2}t^2 + c.$$

By integration by parts we find that

$$\int t^5\ln t\,dt = \frac{1}{6}t^6\ln t - \int \frac{1}{6}t^5\,dt = \frac{t^6}{6}\ln t - \frac{t^6}{36},$$

and so we have

$$t^3x = \frac{t^6}{6}\ln t - \frac{t^6}{36} + \frac{1}{2}t^2 + c.$$

Letting $t = 1$ and $x = 0$ (the initial condition) gives $0 = -\frac{1}{36} + \frac{1}{2} + c$, so that $c = -\frac{17}{36}$ and we obtain

$$x(t) = \frac{1}{6}t^3\left(\ln t - \frac{1}{6}\right) + \frac{1}{2t} - \frac{17}{36t^3}$$

as the solution to the IVP.

9 We have

$$M(x,y) = 1 + \ln y \quad \text{and} \quad N(x,y) = \frac{x}{y}.$$

Since the equation is exact there exists a function $F(x,y)$ such that $F_x = M$ and $F_y = N$; that is,

$$F_x(x,y) = 1 + \ln y \quad \text{and} \quad F_y(x,y) = \frac{x}{y}. \tag{1}$$

Integrate the first equation in (1) with respect to x:

$$F(x, y) = \int (1 + \ln y) dx + g(y) = x(1 + \ln y) + g(y). \tag{2}$$

Differentiating this with respect to y yields

$$F_y(x, y) = \frac{x}{y} + g'(y),$$

and so using the second equation in (1) we obtain

$$\frac{x}{y} + g'(y) = \frac{x}{y},$$

or simply $g'(y) = 0$. Hence $g(y) = c_1$ for some arbitrary constant c_1, and so (2) becomes

$$F(x, y) = x(1 + \ln y) + c_1.$$

The general implicit solution to the ODE is therefore

$$x(1 + \ln y) + c_1 = c_2$$

for arbitrary c_2, which we can write simply as

$$x + x \ln y = c$$

by consolidating the arbitrary constants c_1 and c_2.