Math 250 Exam #5 Key (Spring 2013)

1 Solution is

$$z(t) = e^{-t} + e^{-2t} + \frac{1}{2}\big[e^{-3t} - 2e^{-2(t+1)} + e^{-(t+4)}\big]u(t-2)$$

2 We take the Laplace transform of each side of the ODE, using linearity properties to obtain

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = 2\mathcal{L}[\delta(t-\pi)](s) - \mathcal{L}[\delta(t-2\pi)](s).$$

Now, letting $Y(s) = \mathcal{L}[y](s)$ we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = 2e^{-\pi s} - e^{-2\pi s},$$

whence

$$Y(s) = \frac{2e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4}$$

and so

$$y(t) = \mathcal{L}^{-1}\left[\frac{2e^{-\pi s}}{s^2+4}\right](t) - \mathcal{L}^{-1}\left[\frac{e^{-2\pi s}}{s^2+4}\right](t). \tag{1}$$

If we define $h(t) = \sin(2t)$, then

$$\mathcal{L}[h(t)](s) = \frac{2}{s^2+4},$$

so

$$\mathcal{L}[h(t-\pi)u(t-\pi)](s) = e^{-\pi s}\mathcal{L}[h(t)](s) = \frac{2e^{-\pi s}}{s^2+4}$$

and hence

$$\mathcal{L}^{-1}\left[\frac{2e^{-\pi s}}{s^2+4}\right](t) = h(t-\pi)u(t-\pi) = \sin(2t-2\pi)u(t-\pi) = \sin(2t)u(t-\pi).$$

In similar fashion we obtain

$$\mathcal{L}^{-1}\left[\frac{e^{-2\pi s}}{s^2+4}\right](t) = \frac{1}{2}h(t-2\pi)u(t-2\pi) = \frac{1}{2}\sin(2t-4\pi)u(t-2\pi) = \frac{1}{2}\sin(2t)u(t-2\pi).$$

Putting these results into (1) yields

$$y(t) = \sin(2t)\big[u(t-\pi) - \tfrac{1}{2}u(t-2\pi)\big].$$

3 We have

$$\int_0^t (t-\tau)^2 y(\tau)\,d\tau = (f * g)(t)$$

with $f(t) = t^2$ and $g(t) = y(t)$, so the integral equation may be written as

$$y(t) + (f * y)(t) = t^3 + 3.$$

Taking the Laplace transform of both sides of the equation yields, by the Convolution Theorem,

$$\mathcal{L}[y](s) + \mathcal{L}[f](s)\mathcal{L}[y](s) = \mathcal{L}\big[t^3\big](s) + \mathcal{L}[3](s),$$

or

$$Y(s) + Y(s)\mathcal{L}\big[t^2\big](s) = \mathcal{L}\big[t^3\big](s) + \mathcal{L}[3](s)$$

if we let $Y(s) = \mathcal{L}[y](s)$. Using a table of Laplace transforms yields

$$Y(s) + Y(s) \cdot \frac{2}{s^3} = \frac{6}{s^4} + \frac{3}{s} \quad \Rightarrow \quad Y(s)\left(\frac{2+s^3}{s^3}\right) = \frac{3(2+s^3)}{s^4} \quad \Rightarrow \quad Y(s) = \frac{3}{s},$$

whence

$$y(t) = \mathcal{L}^{-1}\left[\frac{3}{s}\right](t) = 3$$

obtains as the (unique) solution.

4 $y(t) = 1 - \frac{1}{6}t^3 + \frac{1}{180}t^6 + \cdots$

5 Solution will be of the form

$$y(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Substituting this into the ODE gives

$$\sum_{k=1}^{\infty} k c_k x^{k-1} - \sum_{k=0}^{\infty} c_k x^k = 0.$$

Reindexing, we obtain

$$\sum_{k=0}^{\infty} (k+1) c_{k+1} x^k - \sum_{k=0}^{\infty} c_k x^k = 0,$$

or equivalently

$$\sum_{k=0}^{\infty} \left[(k+1)c_{k+1} - c_k\right] x^k = 0.$$

This implies that $(k+1)c_{k+1} - c_k = 0$ for all $k \geq 0$, or $c_{k+1} = c_k/(k+1)$. From this we find that $c_1 = c_0$, $c_2 = c_1/2 = c_0/2!$, $c_3 = c_2/3 = c_0/3!$, and in general $c_k = c_0/k!$. Therefore

$$y(x) = \sum_{k=0}^{\infty} \frac{c_0}{k!} x^k = c_0 + c_0 x + \frac{c_0}{2} x^2 + \frac{c_0}{6} x^3 + \cdots,$$

where c_0 is an arbitrary constant.

6 Solution is

$$y(t) = 2 + t^2 - \frac{5}{12}t^4 + \frac{11}{72}t^6 + \cdots$$