Math 250 Exam #3 Key (Spring 2013)

1a The nonhomogeneity is

$$f(t) = P_m(t)e^{\alpha t} = 4t^2,$$

so $m = 2$ and $\alpha = 0$. We see that α is not a root of the auxiliary equation $r^2 - 3r + 2 = 0$, so $s = 0$ and we have

$$y_p(t) = t^s e^{\alpha t} \sum_{k=0}^{m} A_k t^k = \sum_{k=0}^{2} A_k t^k = A_0 + A_1 t + A_2 t^2.$$

For convenience we may write $y_p(t) = A + Bt + Ct^2$, so that

$$y_p'(t) = B + 2Ct \quad \text{and} \quad y_p''(t) = 2C.$$

Substituting into the ODE yields

$$2C - 3(B + 2Ct) + 2(A + Bt + Ct^2) = 4t^2,$$

or

$$(2C)t^2 + (-6C + 2B)t + (2C - 3B + 2A) = 4t^2.$$

Equating coefficients gives the system

$$\begin{cases} 2C = 4 \\ 2B - 6C = 0 \\ 2A - 3B + 2C = 0 \end{cases}$$

which has solution $(A, B, C) = (7, 6, 2)$, and thus a particular solution to the ODE is $y_p(t) = 2t^2 + 6t + 7$.

1b The nonhomogeneity is

$$f(t) = P_m(t)e^{\alpha t} \sin \beta t = e^t \sin t,$$

so $m = 0$, $\alpha = 1$, and $\beta = 1$. We see that $\alpha + i\beta = 1 + i$ is not a root of the auxiliary equation $r^2 - 3r + 2 = 0$, so $s = 0$ and we have

$$y_p(t) = e^t \cos t \sum_{k=0}^{0} A_k t^k + e^t \sin t \sum_{k=0}^{0} B_k t^k = A_0 e^t \cos t + B_0 e^t \sin t,$$

or simply $y_p(t) = Ae^t \cos t + Be^t \sin t$. Now,

$$y_p'(t) = (-A + B)e^t \sin t + (A + B)e^t \cos t \quad \text{and} \quad y_p''(t) = 2Be^t \cos t - 2Ae^t \sin t,$$

and substitution into the ODE yields

$$e^t(2B \cos t - 2A \sin t) - 3e^t[(B - A) \sin t + (A + B) \cos t] + 2e^t(A \cos t + B \sin t) = e^t \sin t,$$

or simply

$$(A - B) \sin t + (-A - B) \cos t = \sin t.$$

Equating coefficients yields the system

$$\begin{cases} A - B = 1 \\ -A - B = 0 \end{cases}$$

which has solution $(A, B) = \left(\frac{1}{2}, -\frac{1}{2}\right)$, and thus a particular solution to the ODE is $y_p(t) = e^t\left(\frac{1}{2}\cos t - \frac{1}{2}\sin t\right)$.

1c Using the results of parts (a) and (b), by the Principle of Superposition a particular solution is

$$y_p(t) = \tfrac{1}{2}e^t(\cos t - \sin t) + 2t^2 + 6t + 7.$$

1d The auxiliary equation of the corresponding homogeneous equation $y'' - 3y' + 2y = 0$ is $r^2 - 3r + 2 = 0$, which has roots $1, 2$. Thus the general solution to the homogeneous equation is $y_h(t) = c_1e^t + c_2e^{2t}$, and by the Superposition Principle the general solution to $y'' - 3y' + 2y = e^t \sin t + 4t^2$ is

$$y(t) = \tfrac{1}{2}e^t(\cos t - \sin t) + 2t^2 + 6t + 7 + c_1e^t + c_2e^{2t}.$$

2a The auxiliary equation of the corresponding homogeneous equation $y'' - 2y' + y = 0$ is $r^2 - 2r + 1 = 0$, which has 1 as a double root. Thus the homogeneous equation has $y_1(t) = e^t$ and $y_2(t) = te^t$ as linearly independent solutions, and so by the Method of Variation of Parameters we look for a particular solution to $y'' - 2y' + y = t^{-1}e^t$ of the form $y_p(t) = v_1(t)e^t + v_2(t)te^t$. The functions v_1 and v_2 have derivatives that satisfy the system

$$\begin{cases} e^t v_1'(t) + \qquad\quad te^t v_2'(t) = 0 \\ e^t v_1'(t) + (e^t + te^t)v_2'(t) = t^{-1}e^t \end{cases}$$

or more simply

$$\begin{cases} v_1'(t) + \qquad\quad tv_2'(t) = 0 \\ v_1'(t) + (1 + t)v_2'(t) = t^{-1} \end{cases}$$

From the first equation we have $v_1'(t) = -tv_2'(t)$, which we substitute into the second equation to obtain $v_2'(t) = t^{-1}$ and then $v_2(t) = \ln|t|$. Now,

$$v_1'(t) = -tv_2'(t) = -t \cdot t^{-1} = -1,$$

and so $v_1(t) = -t$. Therefore

$$y_p(t) = -te^t + \ln|t| \cdot te^t = te^t(\ln|t| - 1).$$

2b The general solution to the corresponding homogeneous equation $y'' - 2y' + y = 0$ is $y_h(t) = c_1e^t + c_2te^t$, and therefore by the Superposition Principle we have

$$y(t) = te^t(\ln|t| - 1) + c_1e^t + c_2te^t = te^t\ln|t| + c_1e^t + (c_2 - 1)te^t$$

as the general solution to $y'' - 2y' + y = t^{-1}e^t$. If we let $a_1 = c_1$ and $a_2 = c_2 - 1$, then we may write

$$y(t) = te^t\ln|t| + a_1e^t + a_2te^t.$$

2c The nonhomogeneity $f(t) = t^{-1}e^t$ has the factor t^{-1} which is not a polynomial. That is, $f(t)$ is not of the form $P_m(t)e^t$ for any choice of polynomial $P_m(t)$.

3 The model for the mass-spring system is

$$20y'' + 140y' + 200y = 0, \quad y(0) = 0.25, \quad y'(0) = -1.$$

The auxiliary equation is

$$20r^2 + 140r + 200 = 0,$$

or $r^2 + 7r + 10 = 0$, which has roots $-2, -5$. Thus the general solution to the ODE is $y(t) = c_1e^{-2t} + c_2e^{-5t}$. From the initial condition $y(0) = 0.25$ comes $c_1 + c_2 = 0.25$, and from $y'(t) = -2c_1e^{-2t} - 5c_2e^{-5t}$ and the initial condition $y'(0) = -1$ comes $-2c_1 - 5c_2 = -1$. Solving the system

$$\begin{cases} c_1 + c_2 = 0.25 \\ -2c_1 - 5c_2 = -1 \end{cases}$$

yields $c_1 = \frac{1}{12}$ and $c_2 = \frac{1}{6}$. Therefore

$$y(t) = \frac{1}{12}e^{-2t} + \frac{1}{6}e^{-5t}.$$

Since the equation $\frac{1}{12}e^{-2t} + \frac{1}{6}e^{-5t} = 0$ has no real solution, it follows that the object never returns to the equilibrium position.

4 There are two external forces acting on the object O: $F(t)$ and also gravity. Thus the total external force on O at time t is

$$F_{\text{ext}} = mg + F(t) = (8 \text{ kg})(9.8 \text{ m/s}^2) + \cos 2t \text{ N} = 78.4 + \cos 2t \text{ N}.$$

We need to determine the spring constant k. Upon attaching O to the spring, the spring stretches until its tension comes to equal the magnitude of the gravitational force acting on O. We have, by Hooke's Law, with the understanding that *down* is the *positive* direction,

$$-(8 \text{ kg})(9.8 \text{ m/s}^2) = -mg = -F_{\text{gravity}} = F_{\text{spring}} = -ky = -k(1.96 \text{ m}),$$

and so $-1.96k = -78.4$, which yields a spring constant of $k = 40$ N/m. The model for the mass-spring system, $my'' + by' + ky = F_{\text{ext}}$, is thus

$$8y'' + 3y' + 40y = 78.4 + \cos 2t. \tag{1}$$

The form of a particular solution to

$$8y'' + 3y' + 40y = 78.4$$

is $y_{p_1}(t) = A$, where A is some constant. Now, $y'_{p_1}(t) = y''_{p_1}(t) = 0$, and so substitution into $8y'' + 3y' + 40y = 78.4$ gives $40A = 78.4$ and finally $A = 1.96$. Therefore $y_{p_1}(t) = 1.96$.

The form of a particular solution to

$$8y'' + 3y' + 40y = \cos 2t$$

is $y_{p_2}(t) = A\cos 2t + B\sin 2t$. Substituting this for y in the ODE yields

$$8(A\cos 2t + B\sin 2t)'' + 3(A\cos 2t + B\sin 2t)' + 40(A\cos 2t + B\sin 2t) = \cos 2t,$$

which gives

$$(8A+6B)\cos 2t + (-6A+8B)\sin 2t = \cos 2t,$$

and so we must have

$$\begin{cases} 8A+6B=1 \\ -6A+8B=0 \end{cases}$$

Solving this system of equations yields $A = 2/25$ and $B = 3/50$, and therefore

$$y_{p_2}(t) = \frac{2}{25}\cos 2t + \frac{3}{50}\sin 2t.$$

By the Superposition Principle a particular solution to (1) is $y_p = y_{p_1} + y_{p_2}$, or

$$y_p(t) = 1.96 + \frac{2}{25}\cos 2t + \frac{3}{50}\sin 2t, \tag{2}$$

which happens to be the steady-state solution for the mass-spring system. However, it is common practice to take the equilibrium position to be wherever the spring comes to rest once the mass is attached to it. We're given that the spring stretches 1.96 m, so we shift the point where y equals zero down by 1.96 m by subtracting 1.96 from the right-hand side of (2), resulting in

$$y_p(t) = \frac{2}{25}\cos 2t + \frac{3}{50}\sin 2t$$

as the steady-state solution. Alternatively we may write

$$y_p(t) = 0.1\sin(2t+\varphi),$$

where $\varphi = \arctan(4/3) \approx 0.927$.