Math 250 Exam #2 Key (Spring 2013)

1 We have

$$\frac{M_y - N_x}{N}(x) = \frac{2 - 4xy}{2x^2y - x} = -\frac{2}{x},$$

so

$$\mu(x) = \exp\left(\int -\frac{2}{x}dx\right) = e^{-2\ln x} = \frac{1}{x^2}.$$

Multiplying the ODE by $\mu(x)$ yields the exact equation

$$(3 + x^{-2}y) + (2y - x^{-1})y' = 0.$$

There exists a function F such that

$$F_x(x,y) = 3 + x^{-2}y \quad \text{and} \quad F_y(x,y) = 2y - x^{-1}.$$

From the former equation comes

$$F(x,y) = 3x - \frac{y}{x} + g(y),$$

so the latter equation implies

$$-x^{-1} + g'(y) = 2y - x^{-1} \quad \Rightarrow \quad g'(y) = 2y \quad \Rightarrow \quad g(y) = y^2 + c_1,$$

c_1 arbitrary. The general solution to the ODE is $F(x,y) = c_2$, where c_2 is arbitrary. That is,

$$3x - \frac{y}{x} + y^2 = c,$$

where c is the arbitrary constant deriving from $c_2 - c_1$.

2 Multiply the ODE by $x^m y^n$:

$$(x^{m+3}y^{n+2} - 2x^m y^{n+3}) + (x^{m+4}y^{n+1})y' = 0.$$

For exactness we need $M_y = N_x$, or

$$(n+2)x^{m+3}y^{n+1} - 2(n+3)x^m y^{n+2} = (m+4)x^{m+3}y^{n+1}.$$

Matching coefficients of like terms, we find that we must have $n+2 = m+4$ and $-2(n+3) = 0$, which solves to give $m = -5$ and $n = -3$. Thus an integrating factor is $\mu(x,y) = x^{-5}y^{-3}$, and the ODE becomes

$$(x^{-2}y^{-1} - 2x^{-5}) + (x^{-1}y^{-2})y' = 0,$$

which is exact. We now find a function F such that $F_x(x,y) = x^{-2}y^{-1} - 2x^{-5}$ and $F_y(x,y) = x^{-1}y^{-2}$. From the former equation we obtain

$$F(x,y) = \int \left(x^{-2}y^{-1} - 2x^{-5}\right) dx = -x^{-1}y^{-1} + \tfrac{1}{2}x^{-4} + g(y),$$

and from the latter equation comes

$$x^{-1}y^{-2} = F_y(x,y) = x^{-1}y^{-2} + g'(y),$$

or $g'(y) = 0$. Thus $g(y) = c_1$ for some constant c_1, and we have

$$F(x,y) = -x^{-1}y^{-1} + \tfrac{1}{2}x^{-4} + c_1.$$

The implicit solution to the ODE is therefore $F(x,y) = c_2$ for arbitrary constant c_2, which we may write simply as

$$\tfrac{1}{2}x^{-4} - x^{-1}y^{-1} = c$$

by merging constant terms. Another solution is $y \equiv 0$.

3 Rewrite ODE as

$$y' = \frac{y}{x}\Big(\ln\frac{y}{x} + 1\Big).$$

let $v = y/x$, so $y = xv$ and we get $y' = xv' + v$. ODE then becomes

$$xv' + v = v(\ln v + 1).$$

The equation is separable, leading to

$$\int \frac{1}{v\ln v}dv = \int \frac{1}{x}dx,$$

whence

$$\ln|\ln v| = \ln|x| + c \quad\Rightarrow\quad \Big|\ln\frac{y}{x}\Big| = e^{\ln|x|+c},$$

$c \in \mathbb{R}$ arbitary. The original ODE requires $x, y > 0$, so we can write $|\ln(y/x)| = e^{\ln x + c} = e^c x = \hat{c}x$, where $\hat{c} = e^c > 0$ is arbitrary. Now

$$\ln\frac{y}{x} = \pm\hat{c}x = \tilde{c}x,$$

where $\tilde{c} \neq 0$ is arbitrary. Further manipulating yields

$$\frac{y}{x} = e^{\tilde{c}x} \quad\Rightarrow\quad y = xe^{\tilde{c}x}.$$

If we replace $\tilde{c}$ with 0 we get $y = x$, which a quick check shows is also a solution to the ODE. Thus we may replace $\tilde{c} \in (-\infty, 0) \cup (0, \infty)$ by $c \in \mathbb{R}$ and write

$$y(x) = xe^{cx}$$

for the general solution.

4 The ODE is Bernoulli with $n = -2$, $P(x) = 1$, and $Q(x) = e^x$. Letting $v = y^3$, ODE becomes $v' + 3v = 3e^x$. Multiplying by the integrating factor

$$\mu(x) = e^{\int 3dx} = e^{3x}$$

yields

$$e^{3x}v' + 3e^{3x}v = 3e^{4x} \quad\Rightarrow\quad (e^{3x}v)' = 3e^{4x} \quad\Rightarrow\quad e^{3x}v = \tfrac{3}{4}e^{4x} + c \quad\Rightarrow\quad v = \tfrac{3}{4}e^x + ce^{-3x},$$

and finally

$$y^3 = \tfrac{3}{4}e^x + ce^{-3x}.$$

Also $y \equiv 0$ is a solution.

5 Let $x(t)$ be the amount of the drug in the organ (in grams) at time t. We have

$$x'(t) = \left(\frac{0.2 \text{ g}}{\text{cm}^3}\right)\left(\frac{3 \text{ cm}^3}{\text{s}}\right) - \left(\frac{x \text{ g}}{125 \text{ cm}^3}\right)\left(\frac{3 \text{ cm}^3}{\text{s}}\right) = 0.6 - \frac{3x}{125}.$$

Multiplying by the integrating factor $e^{3t/125}$ yields

$$e^{3t/125}x' + \frac{3}{125}xe^{3t/125} = 0.6e^{3t/125} \quad \Rightarrow \quad \left(xe^{3t/125}\right)' = 0.6e^{3t/125}.$$

Integrating, we obtain

$$xe^{3t/125} = 25e^{3t/125} + c.$$

The initial condition $x(0) = 0$ results in $c = 25$, and so

$$x(t) = 25 - 25e^{3t/125}.$$

The concentration of the drug at time t, $C(t)$, is given by $x(t)/125$, or

$$C(t) = 0.2 - 0.2e^{-3t/125}.$$

Setting $C(t) = 0.1$ and solving, we find that at time $t \approx 28.9$ seconds the concentration of the drug will be 0.1 g/cm^3.

6 Let $c_1\varphi + c_2\psi = 0$, which is to say $c_1te^{2t} + c_2e^{2t} = 0$ for all $t \in \mathbb{R}$. Substituting $t = 0$ immediately gives $c_2 = 0$. Substituting $t = 1$ gives $c_1e^2 = 0$, and hence $c_1 = 0$ also. Therefore φ and ψ are linearly independent on $(-\infty, \infty)$.

7 Auxiliary equation is $r^2 - 4r - 5 = 0$, which has solutions $r = -1, 5$. Hence

$$y(t) = c_1e^{-t} + c_2e^{5t}.$$

Now, $y'(t) = -c_1e^{-t} + 5c_2e^{5t}$, and so the initial conditions give

$$c_1e + c_2e^{-5} = 3 \quad \text{and} \quad -c_1e + 5c_2e^{-5} = 9.$$

Solving the system gives $c_1 = e^{-1}$ and $c_2 = 2e^5$, and therefore

$$y(t) = e^{-(t+1)} + 2e^{5(t+1)}.$$

8 Auxiliary equation is $9r^2 - 12r + 4 = 0$, which has double root $r = 2/3$. General solution is therefore

$$y(t) = c_1e^{2t/3} + c_2te^{2t/3}.$$

9 Auxiliary equation is $12r^3 - 28r^2 - 3r + 7 = 0$, which factors by grouping:

$$(3r - 7)(2r - 1)(2r + 1) = 0.$$

Roots are $r = 7/3, 1/2, -1/2$. General solution:

$$y(t) = c_1e^{-t/2} + c_2e^{t/2} + c_3e^{7t/3}.$$

10 Auxiliary equation is $r^2 + 9 = 0$, which has roots $r = \pm 3i$. Thus we have

$$y(t) = c_1 \cos 3t + c_2 \sin 3t$$

With the initial conditions $y(0) = 1$ and $y'(0) = 1$ we find that $c_1 = 1$ and $c_2 = 1/3$, and so

$$y(t) = \cos 3t + \tfrac{1}{3} \sin 3t.$$