Math 250 Exam #5 Key (Spring 2012)

1a. Since $f'(x) = 2\cos 2x$, $f''(x) = -4\sin 2x$, $f'''(x) = -8\cos 2x$, and $f^{(4)}(x) = 16\sin 2x$, we have

$$P_4(0) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 = 2x - \frac{4}{3}x^3.$$

1b. Given $f(x) = \tan x$, we have

$$\begin{aligned}
f'(x) &= \sec^2 x \\
f''(x) &= 2\sec^2 x \tan x \\
f'''(x) &= 2\sec^4 x + 4\sec^2 x \tan^2 x \\
f^{(4)}(x) &= 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x,
\end{aligned}$$

and so $f(\pi/4) = 1$, $f'(\pi/4) = \sec^2(\pi/4) = 2$, $f''(\pi/4) = 4$, $f'''(\pi/4) = 16$, and $f^{(4)}(\pi/4) = 80$. Now,

$$P_4(\tfrac{\pi}{4}) = 1 + 2(x - \tfrac{\pi}{4}) + 2(x - \tfrac{\pi}{4})^2 + \tfrac{8}{3}(x - \tfrac{\pi}{4})^3 + \tfrac{10}{3}(x - \tfrac{\pi}{4})^4.$$

2. Use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{4(x-3)^{n+1}}{(n+1)^2 + 2(n+1)} \cdot \frac{n^2+2n}{4(x-3)^n}\right| \\
&= \lim_{n\to\infty}\frac{n^2+2n}{n^2+4n+3}|x-3| = |x-3|.
\end{aligned}$$

The series converges if $|x-3| < 1$, or $x \in (2,4)$.

When $x = 4$ the series becomes

$$\sum_{k=0}^{\infty}\frac{4}{n^2+2n},$$

which we conclude must converge by using the Direct Comparison Test and the p-series $\sum \frac{1}{n^2}$ (which is known to converge).

When $x = 2$ the series becomes

$$\sum_{k=0}^{\infty}\frac{4(-1)^n}{n^2+2n},$$

which we conclude must converge by the Alternating Series Test.

Therefore the series converges on the interval $[2,4]$.

3. Solution will be of the form

$$y(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Substituting this into the ODE gives

$$\sum_{k=1}^{\infty} kc_k x^{k-1} - \sum_{k=0}^{\infty} c_k x^k = 0.$$

Reindexing, we obtain

$$\sum_{k=0}^{\infty} (k+1)c_{k+1}x^k - \sum_{k=0}^{\infty} c_k x^k = 0,$$

or equivalently

$$\sum_{k=0}^{\infty} \left[(k+1)c_{k+1} - c_k\right] x^k = 0.$$

This implies that $(k+1)c_{k+1} - c_k = 0$ for all $k \geq 0$, or $c_{k+1} = c_k/(k+1)$. From this we find that $c_1 = c_0$, $c_2 = c_1/2 = c_0/2!$, $c_3 = c_2/3 = c_0/3!$, and in general $c_k = c_0/k!$. Therefore

$$y(x) = \sum_{k=0}^{\infty} \frac{c_0}{k!} x^k = c_0 + c_0 x + \frac{c_0}{2} x^2 + \frac{c_0}{6} x^3 + \cdots ,$$

where c_0 is an arbitrary constant.

4. We find a general solution of the form

$$y(x) = \sum_{k=0}^{\infty} c_k x^k,$$

with the series converging on some open interval I containing 0. Substituting this into the ODE yields

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - x^2 \sum_{k=1}^{\infty} kc_k x^{k-1} - x \sum_{k=0}^{\infty} c_k x^k = 0,$$

and thus

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - \sum_{k=1}^{\infty} kc_k x^{k+1} - \sum_{k=0}^{\infty} c_k x^{k+1} = 0.$$

Reindexing so that all series feature x^k, we have

$$\sum_{k=0}^{\infty} (k+1)(k+2)c_{k+2}x^k - \sum_{k=2}^{\infty} (k-1)c_{k-1}x^k - \sum_{k=1}^{\infty} c_{k-1}x^k = 0.$$

Finally we contrive to have the index of each series start at 2 by removing the first two terms of the leftmost series and the first term of the rightmost series:

$$\left[2c_2 + 6c_3 x + \sum_{k=2}^{\infty} (k+1)(k+2)c_{k+2}x^k\right] - \sum_{k=2}^{\infty} (k-1)c_{k-1}x^k - \left[c_0 x + \sum_{k=2}^{\infty} c_{k-1}x^k = 0\right].$$

Hence

$$2c_2 + (6c_3 - c_0)x + \sum_{k=2}^{\infty} \left[(k+1)(k+2)c_{k+2} - (k-1)c_{k-1} - c_{k-1}\right] x^k = 0,$$

which simplifies to become

$$2c_2 + (6c_3 - c_0)x + \sum_{k=2}^{\infty} \left[(k+1)(k+2)c_{k+2} - kc_{k-1}\right] x^k = 0.$$

This implies that $2c_2 = 0$, $6c_3 - c_0 = 0$, and

$$(k+1)(k+2)c_{k+2} - kc_{k-1} = 0$$

for all $k \geq 2$. That is, $c_2 = 0$, $c_3 = c_0/6 = c_0/(2 \cdot 3)$, and

$$c_{k+2} = \frac{k}{(k+1)(k+2)} c_{k-1}$$

for $k = 2, 3, 4, \ldots$. The recursion relation enables us to express all c_k exclusively in terms of c_0 and c_1:

$$c_4 = \frac{2}{3 \cdot 4} c_1 \qquad c_5 = \frac{3}{4 \cdot 5} c_2 = 0$$

$$c_6 = \frac{4}{5 \cdot 6} c_3 = \frac{4}{2 \cdot 3 \cdot 5 \cdot 6} c_0 \qquad c_7 = \frac{5}{6 \cdot 7} c_4 = \frac{2 \cdot 5}{3 \cdot 4 \cdot 6 \cdot 7} c_1$$

$$c_8 = \frac{6}{7 \cdot 8} c_5 = 0 \qquad c_9 = \frac{7}{8 \cdot 9} c_6 = \frac{4 \cdot 7}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} c_0$$

$$c_{10} = \frac{8}{9 \cdot 10} c_7 = \frac{2 \cdot 5 \cdot 8}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} c_1 \qquad c_{11} = \frac{9}{10 \cdot 11} c_8 = 0$$

$$c_{12} = \frac{10}{11 \cdot 12} c_9 = \frac{4 \cdot 7 \cdot 10}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 11 \cdot 12} c_0$$

So we have

$$y(x) = c_0 + c_1 x + \frac{c_0}{2 \cdot 3} x^3 + \frac{2c_1}{3 \cdot 4} x^4 + \frac{4c_0}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{2 \cdot 5 c_1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + \frac{4 \cdot 7 c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \frac{2 \cdot 5 \cdot 8 c_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \frac{4 \cdot 7 \cdot 10 c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 11 \cdot 12} x^{12} + \cdots$$

Setting $c_0 = 0$ and $c_1 = 1$ yields the particular solution

$$\begin{aligned} y_1(x) &= x + \frac{2}{3 \cdot 4} x^4 + \frac{2 \cdot 5}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + \frac{2 \cdot 5 \cdot 8}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \cdots \\ &= x + \frac{2^2}{4!} x^4 + \frac{2^2 \cdot 5^2}{7!} x^7 + \frac{2^2 \cdot 5^2 \cdot 8^2}{10!} x^{10} + \cdots \\ &= x + \sum_{k=1}^{\infty} \frac{2^2 \cdot 5^2 \cdots (3k-1)^2}{(3k+1)!} x^{3k+1}, \end{aligned}$$

and setting $c_0 = 1$ and $c_1 = 0$ yields the particular solution

$$\begin{aligned} y_2(x) &= 1 + \frac{1}{2 \cdot 3} x^3 + \frac{4}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{4 \cdot 7}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \frac{4 \cdot 7 \cdot 10}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 11 \cdot 12} x^{12} + \cdots \\ &= 1 + \frac{1}{3!} x^3 + \frac{4^2}{6!} x^6 + \frac{4^2 \cdot 7^2}{9!} x^9 + \frac{4^2 \cdot 7^2 \cdot 10^2}{12!} x^{12} + \cdots \\ &= 1 + \sum_{k=1}^{\infty} \frac{4^2 \cdot 7^2 \cdots (3k-2)^2}{(3k)!} x^{3k}. \end{aligned}$$

Since $y_1(x)$ and $y_2(x)$ are linearly independent, the general solution to the ODE may be expressed as

$$y(x) = a_0 \left[x + \sum_{k=1}^{\infty} \frac{2^2 \cdot 5^2 \cdots (3k-1)^2}{(3k+1)!} x^{3k+1} \right] + a_1 \left[1 + \sum_{k=1}^{\infty} \frac{4^2 \cdot 7^2 \cdots (3k-2)^2}{(3k)!} x^{3k} \right]$$

for all $x \in I$, where a_0 and a_1 are arbitrary constants.

5. Since $x = 2$ is an ordinary point for the ODE, we expect to find a general solution of the form

$$y(x) = \sum_{k=0}^{\infty} c_k (x-2)^k, \tag{1}$$

with the power series converging on some open interval I containing 2. From (1) comes

$$y'(x) = \sum_{k=1}^{\infty} k c_k (x-2)^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) c_k (x-2)^{k-2},$$

which when substituted into the ODE yields

$$x^2 \sum_{k=2}^{\infty} k(k-1) c_k (x-2)^{k-2} - \sum_{k=1}^{\infty} k c_k (x-2)^{k-1} + \sum_{k=0}^{\infty} c_k (x-2)^k = 0. \tag{2}$$

It will be expedient to express x^2 in terms of $x - 2$. Since $(x-2)^2 = x^2 - 4x + 4$ we have

$$x^2 = (x-2)^2 + 4x - 4 = (x-2)^2 + 4(x-2) + 4,$$

and so (2) becomes

$$[(x-2)^2 + 4(x-2) + 4] \sum_{k=2}^{\infty} k(k-1) c_k (x-2)^{k-2} - \sum_{k=1}^{\infty} k c_k (x-2)^{k-1} + \sum_{k=0}^{\infty} c_k (x-2)^k = 0,$$

and thus

$$\sum_{k=2}^{\infty} k(k-1) c_k (x-2)^k + 4 \sum_{k=2}^{\infty} k(k-1) c_k (x-2)^{k-1} + 4 \sum_{k=2}^{\infty} k(k-1) c_k (x-2)^{k-2}$$
$$- \sum_{k=1}^{\infty} k c_k (x-2)^{k-1} + \sum_{k=0}^{\infty} c_k (x-2)^k = 0.$$

Adding zero terms and reindexing where needed, we obtain

$$\sum_{k=0}^{\infty} k(k-1) c_k (x-2)^k + 4 \sum_{k=0}^{\infty} k(k+1) c_{k+1} (x-2)^k + 4 \sum_{k=0}^{\infty} (k+1)(k+2) c_{k+2} (x-2)^k$$
$$- \sum_{k=0}^{\infty} (k+1) c_{k+1} (x-2)^k + \sum_{k=0}^{\infty} c_k (x-2)^k = 0,$$

or equivalently

$$\sum_{k=0}^{\infty} \left[k(k-1)c_k + 4k(k+1)c_{k+1} + 4(k+1)(k+2)c_{k+2} - (k+1)c_{k+1} + c_k\right](x-2)^k = 0$$

for all $x \in I$. Therefore we have

$$k(k-1)c_k + 4k(k+1)c_{k+1} + 4(k+1)(k+2)c_{k+2} - (k+1)c_{k+1} + c_k = 0$$

for all $k \geq 0$, which rearranges to become

$$c_{k+2} = -\frac{(4k^2+3k-1)c_{k+1} + (k^2-k+1)c_k}{4k^2+12k+8}. \tag{3}$$

Using the recursion relation (3), we obtain

$$c_2 = \frac{c_1 - c_0}{8}$$

and

$$c_3 = -\frac{6c_2 + c_1}{24} = -\frac{1}{4}\left(\frac{c_1 - c_0}{8}\right) - \frac{1}{24}c_1 = \frac{3c_0 - 7c_1}{96}.$$

Hence

$$\begin{aligned} y(x) &= c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 + \cdots \\ &= c_0 + c_1(x-2) + \frac{c_1 - c_0}{8}(x-2)^2 + \frac{3c_0 - 7c_1}{96}(x-2)^3 + \cdots \end{aligned}$$

is a power series expansion about 2 for a general solution to the ODE.