Math 250 Exam #4 Key (Spring 2012)

1. We have from the table

$$\mathcal{L}\big[te^{3t}\big](s) = \frac{1!}{(s-3)^{1+1}} = \frac{1}{(s-3)^2}.$$

2. We have

$$\begin{aligned}\mathcal{L}\big[t^2 - 3t - 3e^{-t}\sin 3t\big](s) &= \mathcal{L}\big[t^2\big](s) - 3\mathcal{L}[t](s) - 3\mathcal{L}\big[e^{-t}\sin 3t\big](s) \\ &= \frac{2!}{(s-0)^{2+1}} - 3\cdot\frac{1!}{(s-0)^{1+1}} - 3\cdot\frac{3}{(s+1)^2+3^2} \\ &= \frac{2}{s^3} - \frac{3}{s^2} - \frac{9}{(s+1)^2+9}\end{aligned}$$

3. Clearly f is continuous $[0,2)\cup(2,10]$, since it behaves as a polynomial function on $[0,2)$ and $(2,10]$. At $t=2$ we evaluate the one-sided limits:

$$\lim_{t\to 2^-} f(t) = \lim t \to 2^- 0 = 0 \quad \text{and} \quad \lim_{t\to 2^+} f(t) = \lim_{t\to 2^+} t = 2.$$

We see, then, that the one-sided limits are not equal but do exist in $\mathbb{R}$, and so f has a jump-discontinuity at 2. So, since f is continuous on $[0,10]$ except at one point where there is a jump-discontinuity, it is by definition piecewise-continuous on $[0,10]$.

4. Use a trigonometric identity for this.

$$\begin{aligned}\mathcal{L}\big[e^{7t}\sin^2 t\big](s) &= \mathcal{L}\big[e^{7t}\cdot\big(\tfrac{1}{2} - \tfrac{1}{2}\cos 2t\big)\big](s) = \tfrac{1}{2}\mathcal{L}\big[e^{7t}\big](s) - \tfrac{1}{2}\mathcal{L}\big[e^{7t}\cos 2t\big](s) \\ &= \frac{1}{2}\cdot\frac{0!}{(s-7)^{0+1}} - \frac{1}{2}\cdot\frac{s-7}{(s-7)^2+2^2} = \frac{1}{2(s-7)} - \frac{s-7}{2(s-7)^2+8}\end{aligned}$$

5. Partial fraction decomposition is necessary: we have

$$\frac{s+11}{(s-1)(s+3)} = \frac{3}{s-1} - \frac{2}{s+3},$$

and so

$$\mathcal{L}^{-1}\left[\frac{s+11}{(s-1)(s+3)}\right](t) = 3\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) - 2\mathcal{L}^{-1}\left[\frac{1}{s+3}\right](t) = 3e^t - 2e^{-3t}.$$

6. We have

$$\mathcal{L}[y''](s) + 2\mathcal{L}[y'](s) + 2\mathcal{L}[y](s) = \mathcal{L}\big[t^2\big](s) + 4\mathcal{L}[t](s).$$

Letting $Y(s) = \mathcal{L}[y(t)](s)$ and using relevant properties gives

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 2Y(s) = \frac{2}{s^3} + \frac{4}{s^2}.$$

The initial conditions come next:

$$s^2Y + 1 + 2sY + 2Y = \frac{2}{s^3} + \frac{4}{s^2}.$$

Solving for Y yields

$$Y = \frac{2}{s^3(s^2+2s+2)} + \frac{4}{s^2(s^2+2s+2)} - \frac{1}{s^2+2s+2}.$$

Now we employ partial fraction decomposition to obtain

$$\begin{aligned}
Y &= \left(\frac{1}{2s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{s/2}{s^2+2s+2}\right) + \left(-\frac{2}{s} + \frac{2}{s^2} + \frac{2s+2}{s^2+2s+2}\right) - \frac{1}{s^2+2s+2} \\
&= -\frac{3/2}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{3s/2+1}{(s+1)^2+1^2} \\
&= -\frac{3/2}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{3s/2+3/2}{(s+1)^2+1} - \frac{1/2}{(s+1)^2+1}.
\end{aligned}$$

Finally, we take the inverse Laplace transform of each side to get

$$\begin{aligned}
y(t) &= -\frac{3}{2}\mathcal{L}^{-1}\left[\frac{1}{s}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s^3}\right](t) + \frac{3}{2}\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+1}\right](t) \\
&\quad - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right](t) \\
&= -\frac{3}{2} + t + \frac{1}{2}t^2 + \frac{3}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t.
\end{aligned}$$

7a. We have $g(t) = 20u(t) - [20 - 20u(3\pi - t)] + [20 - 20u(4\pi - t)]$, which simplifies to

$$g(t) = 20[u(t) + u(3\pi - t) - u(4\pi - t)],$$

or even

$$g(t) = 20[1 + u(3\pi - t) - u(4\pi - t)]$$

if $t \geq 0$ is understood.

7b. Totally trivial. First we need

$$\mathcal{L}[u(a-t)](s) = \int_0^\infty e^{-st}u(a-t)\,dt = \int_0^a e^{-st}\,dt = \frac{1-e^{-as}}{s}.$$

Now,

$$\begin{aligned}
\mathcal{L}[g(t)](s) &= 20\mathcal{L}[1](s) + 20\mathcal{L}[u(3\pi - t)](s) - 20\mathcal{L}[u(4\pi - t)](s) \\
&= \frac{20}{s} + 20\left(\frac{1-e^{-3\pi s}}{s}\right) - 20\left(\frac{1-e^{-4\pi s}}{s}\right).
\end{aligned}$$

7c. Letting $\mathcal{I}(s) = \mathcal{L}[I(t)](s)$, we have

$$\begin{aligned}
[s^2\mathcal{I} - sI(0) - I'(0)] + 2[s\mathcal{I} - I(0)] + 2\mathcal{I} &= \frac{20}{s} + 20\left(\frac{1-e^{-3\pi s}}{s}\right) - 20\left(\frac{1-e^{-4\pi s}}{s}\right) \\
s^2\mathcal{I} - 10s + 2s\mathcal{I} - 20 + 2\mathcal{I} &= \frac{20 - 20e^{-3\pi s} + 20e^{-4\pi s}}{s} \\
(s^2+2s+2)\mathcal{I} &= 10s + 20 + \frac{20 - 20e^{-3\pi s} + 20e^{-4\pi s}}{s},
\end{aligned}$$

and thus

$$\mathcal{I}(s) = \frac{10s+20}{s^2+2s+2} + \frac{20}{s(s^2+2s+2)} - \frac{20}{s(s^2+2s+2)}e^{-3\pi s} + \frac{20}{s(s^2+2s+2)}e^{-4\pi s}. \tag{1}$$

Partial fraction decomposition gives

$$\frac{20}{s(s^2+2s+2)} = \frac{10}{s} - \frac{10s+20}{s^2+2s+2},$$

and so (1) becomes

$$\mathcal{I}(s) = \frac{10}{s} - \left(\frac{10}{s} - \frac{10s+20}{s^2+2s+2}\right)e^{-3\pi s} + \left(\frac{10}{s} - \frac{10s+20}{s^2+2s+2}\right)e^{-4\pi s}.$$

Rewriting this as

$$\begin{aligned}
\mathcal{I}(s) = \frac{10}{s} - 10\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right)e^{-3\pi s} \\
+ 10\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right)e^{-4\pi s}
\end{aligned}$$

and taking the inverse Laplace transform of both sides, we obtain

$$\begin{aligned}
I(t) = 10 - 10u(t-3\pi)\left[1 - e^{3\pi-t}\cos(t-3\pi) - e^{3\pi-t}\sin(t-3\pi)\right] \\
+ 10u(t-4\pi)\left[1 - e^{4\pi-t}\cos(t-4\pi) - e^{4\pi-t}\sin(t-4\pi)\right],
\end{aligned}$$

or equivalently

$$I(t) = 10 - 10u(t-3\pi)\big[1 + (\cos t + \sin t)e^{-(t-3\pi)}\big] + 10u(t-4\pi)\big[1 - (\cos t + \sin t)e^{-(t-4\pi)}\big].$$