Math 250 Exam #2 Key (Spring 2012)

1. Here $M(x,y) = x^4 - x + y$ and $N(x,y) = -x$, so $(M_y - N_x)/N = -2/x$ is independent of y and we can let

$$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N}\,dx\right) = \exp\left(-\int \frac{2}{x}\,dx\right) = e^{-2\ln|x|} = x^{-2}$$

be an integrating factor. Multiplying the ODE by $\mu(x)$ gives

$$\left(x^2 - \frac{1}{x} + \frac{y}{x^2}\right) - \frac{1}{x}y' = 0,$$

which is exact. We set about finding a function F such that $F_x(x,y) = x^2 - 1/x + y/x^2$ and $F_y(x,y) = -1/x$. From the former equation we obtain

$$F(x,y) = \int \left(x^2 - \frac{1}{x} + \frac{y}{x^2}\right) dx = \frac{1}{3}x^3 - \ln|x| - \frac{y}{x} + g(y),$$

and from the latter equation comes

$$-\frac{1}{x} = F_y(x,y) = -\frac{1}{x} + g'(y),$$

or $g'(y) = 0$. Thus $g(y) = \hat{c}$ for some constant $\hat{c}$, and we have

$$F(x,y) = \frac{1}{3}x^3 - \ln|x| - \frac{y}{x} + \hat{c}.$$

The implicit solution to the ODE is therefore $F(x,y) = c$ for arbitrary constant c; that is

$$\frac{1}{3}x^3 - \ln|x| - \frac{y}{x} = c,$$

where $\hat{c}$ becomes absorbed by the arbitrary constant term. Alternate presentations for the solution are $x^4 - 3x\ln|x| - 3y = cx$ and $y = \frac{1}{3}x^4 - x\ln|x| + cx$. ■

2. Multiply the ODE by $x^m y^n$ to get

$$(x^m y^{n+2} + x^{m+1}y^{n+1}) - x^{m+2}y^n y' = 0.$$

For exactness we need $M_y = N_x$, or

$$(n+2)x^m y^{n+1} + (n+1)x^{m+1}y^n = -(m+2)x^{m+1}y^n.$$

Matching coefficients of like terms, we find that we must have $n+2 = 0$ and $n+1 = -(m+2)$, which solves to give $m = -1$ and $n = -2$. Thus an integrating factor is $\mu(x,y) = x^{-1}y^{-2}$, and the ODE becomes $(1/x + 1/y) - (x/y^2)y' = 0$, which is exact. We now find a function F such that $F_x(x,y) = 1/x + 1/y$ and $F_y(x,y) = -x/y^2$. From the former equation we obtain

$$F(x,y) = \int \left(\frac{1}{x} + \frac{1}{y}\right) dx = \ln|x| + \frac{x}{y} + g(y),$$

and from the latter equation comes

$$-\frac{x}{y^2} = F_y(x,y) = -\frac{x}{y^2} + g'(y),$$

or $g'(y) = 0$. Thus $g(y) = \hat{c}$ for some constant $\hat{c}$, and we have $F(x, y) = \ln|x| + x/y + \hat{c}$. The implicit solution to the ODE is therefore $\ln|x| + x/y + \hat{c} = c$ for arbitrary constant, which we may write simply as $\ln|x| + x/y = c$ by merging constant terms. Another solution is $y \equiv 0$. ■

3. First suppose that $y > 0$. Then

$$y' = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{y} = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{\sqrt{y^2}} = \frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^{-2} + 1}.$$

Letting $v = y/x$, the equation becomes $v + xv' = v + \sqrt{v^{-2}+1}$, which is separable and so becomes

$$\int \frac{1}{\sqrt{v^{-2}+1}}\, dv = \int \frac{1}{x}\, dx \tag{1}$$

Now, if $x > 0$, then $v > 0$ also, in which case

$$v\sqrt{v^{-2}+1} = \sqrt{v^2}\sqrt{v^{-2}+1} = \sqrt{v^2+1},$$

and so, making the substitution $u = v^2 + 1$ along the way, we get

$$\begin{aligned}\int \frac{1}{\sqrt{v^{-2}+1}}\, dv &= \int \frac{v}{v\sqrt{v^{-2}+1}}\, dv = \int \frac{v}{\sqrt{v^2+1}}\, dv = \int \frac{1/2}{\sqrt{u}}\, du \\ &= \sqrt{u} + c = \sqrt{v^2+1} + c = \sqrt{y^2/x^2+1} + c.\end{aligned}$$

Putting this into (1) gives us the solution

$$\sqrt{\frac{y^2}{x^2} + 1} = \ln|x| + c. \tag{2}$$

If $x < 0$, then $v < 0$ also, in which case

$$v\sqrt{v^{-2}+1} = -\sqrt{v^2}\sqrt{v^{-2}+1} = -\sqrt{v^2+1}.$$

Performing the same manipulations as before (only with a negative sign attached) results in the solution

$$-\sqrt{\frac{y^2}{x^2} + 1} = \ln|x| + c. \tag{3}$$

Now, if we suppose that $y < 0$, much the same analysis is performed, again broken into the two cases $x > 0$ and $x < 0$. If $x > 0$, the solution (3) results, and if $x < 0$, the solution (2) results. If $\operatorname{sgn}(x)$ and $\operatorname{sgn}(y)$ denote the sign of x and y, respectively, then the general solution can be written as

$$\operatorname{sgn}(x)\operatorname{sgn}(y)\sqrt{y^2/x^2+1} = \ln|x| + c.$$

■

4. The Bernoulli equation has $n = 3$, $P(x) = -1$, and $Q(x) = e^{2x}$. Let $v = y^{-2}$, so that the equation becomes $v' + 2v = -2e^{2x}$ as indicated by the formula in the notes (and book). This is a linear equation with $P(x) = 2$ and $Q(x) = -2e^{2x}$, so an integrating factor is

$$\mu(x) = e^{\int 2\, dx} = e^{2x}.$$

Multiplying $v' + 2v = -2e^{2x}$ by e^{2x} gives $v'e^{2x} + 2ve^{2x} = -2e^{4x}$, which becomes $(ve^{2x})' = -2e^{4x}$, and so by integration we obtain

$$ve^{2x} = \int -2e^{4x}\,dx = -\frac{1}{2}e^{4x} + c.$$

Hence $v = -\frac{1}{2}e^{2x} + ce^{-2x}$, so that $y^{-2} = -\frac{1}{2}e^{2x} + ce^{-2x}$. That is, the general solution to the ODE is

$$y^2 = \frac{2}{ce^{-2x} - e^{2x}}.$$

Another solution happens to be $y \equiv 0$. ■

5. Let $x(t)$ be the number of gallons of Cl in the pool at time t, so $x(0) = 1$ (0.01% of 10,000). Now, 5 gallons of solution that is 0.001% Cl by volume is coming in per minute, which is to say that 0.00005 gallons of Cl is entering per minute. Meanwhile there are $x(t)/10,000$ gallons of Cl per gallon of solution in the pool, and this solution is being pumped out at a rate of 5 gallons per minute. Thus, $5x(t)/10,000$ gallons of Cl is leaving per minute. We have

$$x'(t) = 0.00005 - \frac{5x(t)}{10,000} = \frac{0.1 - x}{2000}$$

This equation is separable, and so becomes

$$\int \frac{2000}{0.1 - x}\,dx = \int dt,$$

and hence $-2000 \ln|x - 0.1| = t + c$. Solving for x gives $x(t) = 0.1 + Ke^{-t/2000}$, and using the initial condition $x(0) = 1$ we find that $K = 0.9$. So finally we have $x(t) = 0.1 + 0.9e^{-t/2000}$.

The amount of Cl in the pool after 60 minutes (1 hour) is $x(60) = 0.1 + 0.9e^{-60/2000} = 0.973$ gallons, which means the pool is 0.00973% Cl.

We now find the time t when the pool is 0.002% Cl, or in other words $x(t) = 0.2$ gallons Cl. The equation is $0.2 = 0.1 + 0.9e^{-t/2000}$, which solves to give $t = 4394.4$ minutes, or 73.24 hours. ■

6. Auxiliary equation is $r^2 - 4r - 5 = 0$, so $r = -1, 5$ and the general solution is $y(t) = c_1e^{-t} + c_2e^{5t}$. Using the initial conditions $y(-1) = 3$ and $y'(-1) = 9$, we find that $c_1 = e^{-1}$ and $c_2 = 2e^5$. Thus the solution to the IVP is $y(t) = e^{-t-1} + 2e^{5t+5}$, or

$$y(t) = e^{-(t+1)} + 2e^{5(t+1)}$$

will also do. ■

7. Auxiliary equation is $r^3 - 6r^2 - r + 6 = 0$, which factors as $(r - 6)(r^2 - 1) = 0$ and finally $(r - 6)(r - 1)(r + 1) = 0$. Thus $r = 6, 1, -1$, and the general solution is

$$y(t) = c_1e^{6t} + c_2e^t + c_3e^{-t},$$

a three-parameter family of functions. ■

8. Auxiliary equation is $r^2 - 2r + 26 = 0$, which has roots $r = 1 \pm 5i$. So $\alpha = 1$ and $\beta = 5$, and the general solution is therefore $y(t) = e^t(c_1 \cos 5t + c_2 \sin 5t)$. ■