Math 250 Exam #1 Key (Spring 2012)

1a. If the function $x(t)$ gives position x at time t, then $x'(t)$ gives velocity at time t. Then $x'(t) = kx^2(t)$, or more simply $x' = kx^2$, is the equation, where k is some constant of proportionality. ■

1b. The function $A(t)$ gives the mass A of salt at time t, and so

$$A'(t) = \frac{k}{\sqrt{A(t)}}$$

is the appropriate ODE. ■

2. Substitution gives $(ce^{3x}+1)' - 3(ce^{3x}+1) = -3 \;\Rightarrow\; 3ce^{3x} - 3ce^{3x} - 3 = 3 \;\Rightarrow\; -3 = -3$, which confirms that $\varphi(x) = ce^{3x} + 1$ is a solution for any $c \in \mathbb{R}$. ■

3. Substitute $\varphi(x)$ for y to get

$$x^2\varphi''(x) - x\varphi'(x) - 5\varphi(x) = 0.$$

Since $\varphi(x) = x^m$ we have $\varphi'(x) = mx^{m-1}$ and $\varphi''(x) = m(m-1)x^{m-2}$, so that the equation becomes

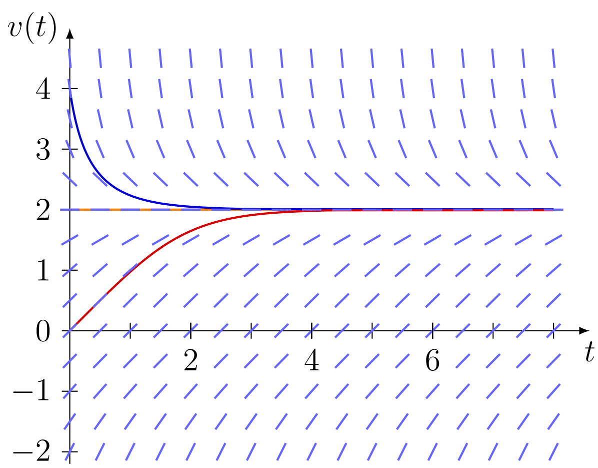
$$x^2 \cdot m(m-1)x^{m-2} - x \cdot mx^{m-1} - 5x^m = 0,$$

and hence

$$m(m-1)x^m - mx^m - 5x^m = 0.$$

Factoring out x^m then yields $[m(m-1) - m - 5]x^m$, or simply $(m^2 - 2m - 5)x^m = 0$. This will be satisfied on an interval I for x only if $m^2 - 2m - 5 = 0$, which by the quadratic formula gives $m = 1 \pm \sqrt{6}$. ■

4. The solution curves corresponding to the initial conditions $v(0) = 0$, $v(0) = 2$, and $v(0) = 4$ are below. It can be seen that $v(t) \to 2$ as $t \to \infty$.



5. We are given $(x_0, y_0) = (1, 0)$ and $h = 0.1$.

n	0	1	2	3	4	5
x_n	1.0	1.1	1.2	1.3	1.4	1.5
y_n	0.0000	0.1000	0.2090	0.3246	0.4441	0.5644

6. This is a separable equation. From $\int e^{2y}\,dy = \int 8x^3\,dx$ we obtain $\frac{1}{2}e^{2y} = 2x^4 + c$. Substituting $x = 1$ and $y = 0$ (the initial condition) gives $\frac{1}{2} = 2 + c$, or $c = -\frac{3}{2}$. Thus the (implicit) solution to the IVP is $\frac{1}{2}e^{2y} = 2x^4 - \frac{3}{2}$, or $e^{2y} = 4x^4 - 3$. The explicit solution may be written $y(x) = \ln\sqrt{4x^4 - 3}$. ■

7. The equation may be written as $y' + \dfrac{3}{x}y = \dfrac{\sin x}{x^2} - 3x$, which is the standard form for a 1st-order linear ODE. An integrating factor is

$$\mu(x) = e^{\int 3/x\,dx} = e^{3\ln x} = x^3.$$

Multiplying the ODE by x^3 gives $x^3y' + 3x^2y = x\sin x - 3x^4$, which becomes $(x^3y)' = x\sin x - 3x^4$ and thus

$$x^3y = \int x\sin x\,dx - \frac{3}{5}x^5 + c.$$

Integration by parts gives

$$\int x\sin x\,dx = \sin x - x\cos x,$$

so that $x^3y = \sin x - x\cos x - \frac{3}{5}x^5 + c$ and therefore

$$y(x) = \frac{1}{x^3}\left(\sin x - x\cos x - \frac{3}{5}x^5 + c\right).$$

is the general solution. ■

8. The equation may be written as

$$x' + \frac{3}{t}x = t^2\ln t + \frac{1}{t^2},$$

which is the standard form for a 1st-order linear ODE. An integrating factor is

$$\mu(x) = e^{\int 3/t\,dt} = t^3.$$

Multiplying the ODE by t^3 gives $t^3x' + 3t^2x = t^5\ln t + t$, which becomes $(t^3x)' = t^5\ln t + t$ and thus

$$t^3x = \int t^5\ln t\,dt + \frac{1}{2}t^2 + c.$$

By integration by parts we find that

$$\int t^5\ln t\,dt = \frac{1}{6}t^6\ln t - \int \frac{1}{6}t^5\,dt = \frac{t^6}{6}\ln t - \frac{t^6}{36},$$

and so we have

$$t^3x = \frac{t^6}{6}\ln t - \frac{t^6}{36} + \frac{1}{2}t^2 + c.$$

Letting $t = 1$ and $x = 0$ (the initial condition) gives $0 = -\frac{1}{36} + \frac{1}{2} + c$, so that $c = -\frac{17}{36}$ and we obtain

$$x(t) = \frac{1}{6}t^3\left(\ln t - \frac{1}{6}\right) + \frac{1}{2t} - \frac{17}{36t^3}$$

as the solution to the IVP. ■

9. The equation is given to be exact, with

$$M(x, y) = ye^{xy} - \frac{1}{y}$$

and

$$N(x, y) = xe^{xy} + \frac{x}{y^2}.$$

We find a function F such that $F_x = M$ and $F_y = N$. Now,

$$F(x, y) = \int F_x(x, y)\,dx = \int M(x, y)\,dx = \int \left(ye^{xy} - \frac{1}{y}\right)dx = e^{xy} - \frac{x}{y} + g(y).$$

Differentiating with respect to y then yields

$$F_y(x, y) = xe^{xy} + \frac{x}{y^2} + g'(y),$$

where $F_y = N$ implies that

$$xe^{xy} + \frac{x}{y^2} + g'(y) = xe^{xy} + \frac{x}{y^2},$$

so that $g'(y) = 0$ and hence $g(y) = c_1$ for some arbitrary constant c_1. This leaves us with

$$F(x, y) = e^{xy} - \frac{x}{y} + c_1.$$

The general implicit solution to the ODE is given by $F(x, y) = c_2$ for arbitrary constant c_2, which here becomes

$$e^{xy} - \frac{x}{y} + c_1 = c_2.$$

Letting $c = c_2 - c_1$, we finally write $ye^{xy} - x = cy$. ■