MATH 250 FINAL EXAM KEY (SPRING 2003)

1 General algorithm: for $y' = f(x,y)$, $y(x_0) = y_0$, let $x_{n+1} = x_n + h$ & $y_{n+1} = y_n + hf(x_n, y_n)$, $n = 0, 1, 2, \ldots$

Here, $y' = 2x + y$, $y(0) = 1$, and $h = 0.1$

So $x_0 = 0$ & $y_0 = 1$

Also $x_1 = 0.1$, $x_2 = 0.2, \ldots, x_5 = 0.5$

And $f(x,y) = 2x + y \Rightarrow f(x_n, y_n) = 2x_n + y_n$

Thus $y_1 = y_0 + 0.1 f(x_0, y_0) = 1 + 0.1[2(0) + 1] \Rightarrow$

- $y_1 = 1 + 0.1 = 1.1$
- $y_2 = y_1 + 0.1 f(x_1, y_1) = 1.1 + 0.1[2(0.1) + 1.1] = 1.23$
- $y_3 = 1.23 + 0.1[2(0.2) + 1.23] = 1.393$
- $y_4 = 1.393 + 0.1[2(0.3) + 1.393] = 1.5923$
- $y_5 = 1.5923 + 0.1[2(0.4) + 1.5923] = 1.83153$
- $y_6 = 1.83153 + 0.1[2(0.5) + 1.83153] = 2.114683$

2 Standard Form: $\frac{dy}{dx} + \frac{3}{x}y = 3x - 2$, so

$P(x) = \frac{3}{x}$ & $Q(x) = 3x - 2$

$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$

So $y(x) = x^{-3}\left[\int x^3(3x-2)\,dx + C\right]$

$= x^{-3}\int (3x^4 - 2x^3)\,dx + Cx^{-3}$

$= x^{-3}\left[\frac{3}{5}x^5 - \frac{2}{4}x^4\right] + Cx^{-3} \Rightarrow$

$$\underline{\underline{y(x) = \frac{3}{5}x^2 - \frac{1}{2}x + Cx^{-3}}}$$

3 $\frac{dy}{d\theta} = \sec(y/\theta) + y/\theta$

Let $v = \frac{y}{\theta}$, so $y = v\theta \Rightarrow$

$\frac{dy}{d\theta} = v + \theta\frac{dv}{d\theta}$

Then equation becomes $v + \theta\frac{dv}{d\theta} = \sec(v) + v \Rightarrow$

$\theta\frac{dv}{d\theta} = \sec(v) \Rightarrow \frac{dv}{\sec(v)} = \frac{d\theta}{\theta} \Rightarrow$

$\int \cos(v)\,dv = \int \frac{1}{\theta}d\theta \Rightarrow \sin(v) = \ln|\theta| + C \Rightarrow$

$\sin(y/\theta) - \ln|\theta| = C$

(Or $e^{\sin(y/\theta)} - K\theta = 0$, $K \neq 0$)

4 $\frac{dm}{dt} \propto m \Rightarrow \frac{dm}{dt} = km \Rightarrow k\,dt = \frac{dm}{m} \Rightarrow$

$t = \frac{1}{k}\int \frac{1}{m}dm = \frac{1}{k}(\ln|m| + c) \Rightarrow$

$t = \frac{1}{k}\ln m + a \Rightarrow e^t = e^{(\ln m)/k + a} = bm^{1/k} \Rightarrow$

$m^{1/k} = \frac{1}{b}e^t \Rightarrow m = de^{kt}$, where $d \equiv \frac{1}{b^k}$

Since $m(0) = 300\text{g}$, we have $300 = de^0 = d$

So $m(t) = 300e^{kt}$

Since $m(5) = 200\text{g}$, we have $200 = 300e^{5k} \Rightarrow$

$5k = \ln(2/3) \Rightarrow k = 0.2\ln(2/3) \approx -0.08109$

$\therefore m(t) = 300e^{-0.08109t}$

Let $m = 1\text{g}$, so...

$1 = 300e^{-0.08109t} \Rightarrow -0.08109t = \ln(1/300) \Rightarrow$

$$\underline{\underline{t \approx 70.34 \text{ years}}}$$

5 Characteristic equation: $r^4 + 13r^2 + 36 = 0 \Rightarrow$

$(r^2+9)(r^2+4) = 0 \Rightarrow r^2 = \{-4, -9\} \Rightarrow$

$r = \{\pm 2i, \pm 3i\}$

Thus, by superposition,

$$\underline{\underline{y(x) = C_1\cos 2x + C_2\sin 2x + C_3\cos 3x + C_4\sin 3x}}$$

6 Solve $y_h'' + 2y_h' + y_h = 0$ for y_h first. Characteristic equation is $r^2 + 2r + 1 = 0 \Rightarrow r = \{-1, -1\}$

So $y_h(\theta) = C_1e^{-\theta} + C_2\theta e^{-\theta}$

Thus it should suffice to suppose that $y_p(\theta) = A\cos\theta + B\sin\theta$ is the form for the particular solution to the original nonhomogeneous diff. eq., since it's linearly independent from the general homogeneous solution. This yields...

$(A\cos\theta + B\sin\theta)'' + 2(A\cos\theta + B\sin\theta)' + (A\cos\theta + B\sin\theta) = 2\cos\theta$

$\cancel{-A\cos\theta} \cancel{- B\sin\theta} - 2A\sin\theta + 2B\cos\theta + \cancel{A\cos\theta} + \cancel{B\sin\theta} = 2\cos\theta$

$2B\cos\theta - 2A\sin\theta = 2\cos\theta$

Thus $2B = 2$ & $-2A = 0 \Rightarrow B = 1$ & $A = 0$

$\therefore y_p(\theta) = \sin\theta$

$\therefore y(\theta) = C_1e^{-\theta} + C_2\theta e^{-\theta} + \sin\theta$

Then $\underline{\underline{y(\theta) = 3e^{-\theta} + 2\theta e^{-\theta} + \sin\theta}}$

$\boxed{7}$ $\mathcal{L}\{y'''\} + 4\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = -12\mathcal{L}\{1\} \Rightarrow$

$$[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)] + 4[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] - 6Y(s) = -\frac{12}{s} \Rightarrow$$

$$s^3Y(s) - s^2 - 4s + 2 + 4[s^2Y(s) - s - 4] + sY(s) - 1 - 6Y(s) = -\frac{12}{s} \Rightarrow$$

$$s^3Y(s) - s^2 - 4s + 2 + 4s^2Y(s) - 4s - 16 + sY(s) - 1 - 6Y(s) = -\frac{12}{s} \Rightarrow$$

$$(s^3 + 4s^2 + s - 6)Y(s) - s^2 - 8s - 15 = -\frac{12}{s}$$

$$Y(s) = \frac{s^2+8s+15}{s^3+4s^2+s-6} - \frac{12}{s(s^3+4s^2+s-6)} = \frac{(s+3)(s+5)}{(s+3)(s+2)(s-1)} - \frac{12}{s(s+3)(s+2)(s-1)} \Rightarrow$$

$$Y(s) = \frac{s+5}{(s+2)(s-1)} - \frac{12}{s(s+3)(s+2)(s-1)}$$

- $\frac{s+5}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} \Rightarrow s+5 = A(s-1) + B(s+2) \Rightarrow (A+B)s + (-A+2B) = s+5$

  So $\begin{cases} A+B=1 \\ -A+2B=5 \end{cases} \Rightarrow 3B = 6 \Rightarrow \underline{B=2}$ & $A = 1-B \Rightarrow \underline{A=-1}$ ✓

- $\frac{12}{s(s+3)(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+2} + \frac{D}{s-1} \Rightarrow 12 = A(s+3)(s+2)(s-1) + Bs(s+2)(s-1)$

  $+ Cs(s+3)(s-1) + Ds(s+3)(s+2) \Rightarrow$

$$A(s^3+4s^2+s-6) + B(s^3+s^2-2s) + C(s^3+2s^2-3s) + D(s^3+5s^2+6s) = 12$$
$$(A+B+C+D)s^3 + (4A+B+2C+5D)s^2 + (A-2B-3C+6D)s - 6A = 12$$

So $-6A = 12 \Rightarrow \underline{A=-2}$

Thus... $\begin{cases} B+C+D=2 \Rightarrow B = 2-C-D \\ B+2C+5D=8 \Rightarrow 2-C-D+2C+5D=8 \\ -2B-3C+6D=2 \Rightarrow -4+2C+2D-3C+6D=2 \end{cases} \longrightarrow \begin{cases} C+4D=6 \\ -C+8D=6 \end{cases} \Rightarrow 12D = 12 \Rightarrow \underline{D=1}$

Then $C = 8D-6 = 8-6 = 2 \Rightarrow \underline{C=2}$

Also $B = 2-C-D = 2-2-1 = -1 \Rightarrow \underline{B=-1}$ ✓

$$\therefore Y(s) = \frac{2}{s-1} - \frac{1}{s+2} + \frac{2}{s} + \frac{1}{s+3} - \frac{2}{s+2} - \frac{1}{s-1} = \frac{1}{s-1} - \frac{3}{s+2} + \frac{1}{s+3} + \frac{2}{s}$$

Thus $y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \Rightarrow$

$$\underline{\underline{y(t) = e^{t} - 3e^{-2t} + e^{-3t} + 2}}$$

$\boxed{8}$ $y' = 2xy - y^2$, $y(0) = 3$

$y'(0) = 2(0)(3) - 3^2 = -9$

$y'' = 2xy' + 2y - 2yy'$

$y''(0) = 2(0)(-9) + 2(3) - 2(3)(-9) = 6 + 54 = 60$

$y''' = 2xy'' + 2y' + 2y' - 2yy'' - 2y'y' = 2xy'' + 4y' - 2yy'' - 2(y')^2$

$y'''(0) = 4(-9) - 2(3)(60) - 2(-9)^2 = -558$

$\therefore\ y(x) \approx \underline{\underline{p_3(x) = 3 - 9x + 30x^2 - 93x^3}}$

$\boxed{9}$ Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$, so differential equation becomes...

$$(2x-3)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\underbrace{2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}}_{\text{Let } k=n-1} - \underbrace{3\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}_{\text{Let } k=n-2} - \underbrace{\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n}_{\text{Let } k=n} = 0$$

$$2\sum_{k=1}^{\infty}(k+1)k a_{k+1} x^k - 3\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$2\sum_{k=1}^{\infty}(k+1)k a_{k+1} x^k - 6a_2 - 3\sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} k a_k x^k + a_0 + \sum_{k=1}^{\infty} a_k x^k = 0$$

$$a_0 - 6a_2 + \sum_{k=1}^{\infty}\left[2(k+1)k a_{k+1} - 3(k+1)(k+2)a_{k+2} - k a_k + a_k\right]x^k = 0$$

Must have $a_0 - 6a_2 = 0 \Rightarrow 6a_2 = a_0 \Rightarrow a_2 = \frac{1}{6}a_0$

Also, $(4a_2 - 18a_3)x = 0 \Rightarrow 4a_2 - 18a_3 = 0 \Rightarrow 18a_3 = 4a_2 \Rightarrow a_3 = \frac{2}{9}a_2 = \frac{2}{9}\left(\frac{1}{6}a_0\right) = \frac{1}{27}a_0$

$(12a_3 - 36a_4 - a_2)x^2 = 0 \Rightarrow 12a_3 - 36a_4 - a_2 = 0 \Rightarrow 36a_4 = 12a_3 - a_2 \Rightarrow$

$a_4 = \frac{1}{3}a_3 - \frac{1}{36}a_2 = \frac{1}{3}\left(\frac{1}{27}a_0\right) - \frac{1}{36}\left(\frac{1}{6}a_0\right) = \frac{1}{81}a_0 - \frac{1}{216}a_0 = \frac{5}{648}a_0$

$$y(x) = a_0\left(1 + \frac{1}{6}x^2 + \cdots\right) + a_1\left(x + \frac{1}{27}x^3 + \cdots\right)$$