MATH 250 EXAM #2 KEY (SPRING 2003)

1 $\frac{\partial}{\partial x}\overbrace{(e^t x+1)}^{M(t,x)} = e^t$, $\frac{\partial}{\partial t}\overbrace{(e^t-1)}^{N(t,x)} = e^t$

Exact ✓

Need a function $F(t,x)$ such that $\frac{\partial F}{\partial t} = M(t,x)$ and $\frac{\partial F}{\partial x} = N(t,x)$. That way we get:

$M(t,x)dt + N(t,x)dx = 0 \Rightarrow \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dx = 0$

$\Rightarrow \partial F + \partial F = 0 \Rightarrow \partial F(t,x) = 0 \Rightarrow F(t,x) = C$.

- Start with $\frac{\partial F}{\partial t} = M(t,x) \Rightarrow dF = (e^t x+1)dt \Rightarrow$

  $F(t,x) = \int (e^t x+1)dt = e^t x + t + g(x)$

- Using $\frac{\partial F}{\partial x} = N(t,x) \Rightarrow e^t + g'(x) = e^t - 1 \Rightarrow$

  $g'(x) = -1 \Rightarrow g(x) = -x$

- Then solution is $F(t,x) = C \Rightarrow$

  $e^t x + t - x = C$

- Since $x(1)=1$, $e^1 + 1 - 1 = C \Rightarrow C = e$
- So $e^t x + t - x = e \Rightarrow x(e^t - 1) = e - t$

$\therefore x(t) = \frac{e-t}{e^t-1}$

2 $2xy\,dy = -(x^2+y^2)dx \Rightarrow$

$\frac{dy}{dx} = \frac{-(x^2+y^2)}{2xy} = \frac{-x^2}{2xy} - \frac{y^2}{2xy}$

$\frac{dy}{dx} = -\frac{x}{2y} - \frac{y}{2x} = -\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)^{*}$

Let $v = \frac{y}{x}$, so $y = vx \Rightarrow$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$*$ So we get $v + x\frac{dv}{dx} = -\frac{1}{2}\left(\frac{1}{v}+v\right) \Rightarrow$

$x\frac{dv}{dx} = -\frac{1}{2v} - \frac{v}{2} - v = -\frac{1}{2v} - \frac{3v}{2}$

$x\frac{dv}{dx} = -\frac{1+3v^2}{2v} \Rightarrow$

$-\frac{2v}{1+3v^2}dv = \frac{1}{x}dx \Rightarrow$

$2\int \frac{v}{1+3v^2}dv = -\int \frac{1}{x}dx$

Let $\beta = 1+3v^2 \Rightarrow \frac{d\beta}{dv} = 6v \Rightarrow \frac{1}{6}d\beta = v\,dv$

Then $\frac{2}{6}\int \frac{1}{\beta}d\beta = -\ln|x| + C \Rightarrow$

$\frac{1}{3}\ln|\beta| = -\ln|x| + C \Rightarrow$

$\ln|\beta|^{1/3} = \ln|x|^{-1} + C \Rightarrow$

$|\beta|^{1/3} = e^{\ln|x|^{-1}+C} = K|x|^{-1}$, where $K \equiv e^C$

$|1+3v^2|^{1/3} = \frac{K}{|x|} \Rightarrow \left(1+\frac{3y^2}{x^2}\right)^{1/3} = \frac{K}{x} \Rightarrow$

$1+\frac{3y^2}{x^2} = \frac{K}{x^3} \Rightarrow x^3 + 3xy^2 = K$

3 Let $z = 2x+y$, so $y = z - 2x \Rightarrow$

$\frac{dy}{dx} = \frac{dz}{dx} - 2$

Then eqn. becomes $\frac{dz}{dx} - 2 = (z-1)^2 =$

$\frac{dz}{dx} = (z-1)^2 + 2 \Rightarrow \frac{1}{(z-1)^2+2}dz = dx$

Let $u = z-1 \Rightarrow \frac{du}{dz} = 1 \Rightarrow du = dz$

Then we have $\int \frac{1}{(\sqrt{2})^2+u^2}du = \int dx \Rightarrow$

$\frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right) = x + C \Rightarrow$

$\arctan\left(\frac{2x+y-1}{\sqrt{2}}\right) = \sqrt{2}x + C \Rightarrow$

$\frac{2x+y-1}{\sqrt{2}} = \tan(\sqrt{2}x + C) \Rightarrow$

$y = 1 - 2x + \sqrt{2}\tan(\sqrt{2}x + C)$

4 $p(t) \equiv$ percent CO at time t

$p(0) = 3$. Find t for when $p = 0.01$

Volume of room $= 12\times 8\times 8 = 768\ ft^3$

- Now define $v(t) =$ volume CO at time t

  Then $v(0) = 0.03(768\ ft^3) = 23.04\ ft^3$.

  Find t for when $v = 0.0001\ (768\ ft^3) = 0.0768\ ft^3$

- $\frac{dv}{dt} =$ volume entering $-$ volume leaving

$= 0\ ft^3\ CO/min. - \left[\frac{v(t)\ ft^3\ CO}{768\ ft^3\ air}\right](100\ ft^3\ air/min) \Rightarrow$

$\frac{dv}{dt} = -\frac{v}{7.68} \Rightarrow \frac{1}{v}dv = -\frac{1}{7.68}dt \Rightarrow$

$\ln v = -\frac{t}{7.68} + C \Rightarrow v(t) = Ke^{-t/7.68}$

Since $v(0) = 23.04$, $23.04 = Ke^0 \Rightarrow K = 23.04$

So $v(t) = 23.04e^{-t/7.68}$

Then $23.04e^{-t/7.68} = 0.0768 \Rightarrow$

$e^{-t/7.68} = 0.00\overline{3} \Rightarrow t = 43.81$ min.

$\boxed{5}$ Divide by x^2 to get:

$$z'' + \frac{1}{x}z' + \frac{1}{x^2}z = \frac{\cos x}{x^2}, \quad z(\underset{x_0}{0}) = 1.$$

Here, $p(x) = \frac{1}{x}$, which is not continuous on any interval (a,b) that contains the point $x_0 = 0$. For this reason the theorem does not apply.

$\boxed{6a}$ $y_1(x) = 2x^3$ & $y_2(x) = x^3$ $\forall x \in [0,\infty)$

Clearly $C_1y_1(x) + C_2y_2(x) = 0$ $\forall x \in [0,\infty)$ if $C_1 = 1$ and $C_2 = -2$.

$\therefore$ y_1 & y_2 are linearly dependent

$\boxed{6b}$ $y_1(x) = 2x^3$ & $y_2(x) = -x^3$ $\forall x \in (-\infty, 0]$

Here $C_1y_1 + C_2y_2 = 0$ $\forall x \in (-\infty, 0]$ if $C_1 = 1$ and $C_2 = 2$

$\therefore$ linearly dependent

$\boxed{6c}$ $y_1(x) = 2x^3$ & $y_2(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$

- For $x \geq 0$, $C_1y_1 + C_2y_2 = 0$ if $C_1 =$ anything and $C_2 = -2C_1$:
  $C_1y_1 + C_2y_2 = C_1 \cdot 2x^3 - 2C_1 \cdot x^3 \equiv 0$ $\forall x \geq 0$
- But this does not work for $x < 0$:
  $C_1y_1 + C_2y_2 = C_1 \cdot 2x^3 - 2C_1(-x^3)$
  $= 2C_1x^3 + 2C_1x^3 \equiv 0$ $\forall x < 0$ only if $C_1 = 0$, which means $C_2 = -2C_1 = -2 \cdot 0 = 0$.

$\therefore$ linearly independent.

$\boxed{6d}$ For $x \geq 0$, $W[y_1, y_2](x) = y_1y_2' - y_1'y_2$

$= 2x^3 \cdot 3x^2 - 6x^2 \cdot x^3 = 6x^5 - 6x^5 = 0$

For $x < 0$, $W[y_1, y_2](x)$

$= 2x^3 \cdot (-3x^2) - 6x^2 \cdot (-x^3)$

$= -6x^5 + 6x^5 = 0$

$\therefore W[y_1, y_2](x) = 0$ $\forall x \in (-\infty, \infty)$

$\boxed{7}$ $L[\ln x] = (D^3 - x^2D^2 + 4xD)[\ln x]$

$$= \frac{\partial^3}{\partial x^3}(\ln x) - x^2\frac{\partial^2}{\partial x^2}(\ln x) + 4x\frac{\partial}{\partial x}(\ln x)$$

$$= \frac{2}{x^3} - x^2\left(-\frac{1}{x^2}\right) + 4x\left(\frac{1}{x}\right) = \frac{2}{x^3} + 5 \checkmark$$

$\boxed{8}$ $T[cy] = (cy)'' + [(cy)'(cy)^2]^{1/3}$

$= cy'' + (cy' \cdot c^2y^2)^{1/3}$

$= cy'' + c(y'y^2)^{1/3} = cT[y] \checkmark$

$T[y_1 + y_2] = (y_1 + y_2)'' + [(y_1 + y_2)'(y_1 + y_2)^2]^{1/3}$

$= y_1'' + y_2'' + [(y_1' + y_2')(y_1^2 + 2y_1y_2 + y_2^2)]^{1/3}$

$T[y_1] + T[y_2] = y_1'' + (y_1'y_1^2)^{1/3} + y_2'' + (y_2'y_2^2)^{1/3}$

$= y_1'' + y_2'' + [(y_1'y_1^2)^{1/3} + (y_2'y_2^2)^{1/3}]$

Clearly $T[y_1 + y_2] \neq T[y_1] + T[y_2] \Rightarrow$ Not linear

$\boxed{9}$ Characteristic equation is $r^2 - 6r + 9 = 0 \Rightarrow$

$(r-3)^2 = 0 \Rightarrow r = \{3, 3\}$

Thus $y(x) = C_1e^{3x} + C_2xe^{3x}$

$y(0) = 2 \Rightarrow 2 = C_1$

Also $y'(x) = 3C_1e^{3x} + 3C_2xe^{3x} + C_2e^{3x}$

$y'(0) = 25/3 \Rightarrow 25/3 = 3 \cdot 2e^0 + 0 + C_2e^0 \Rightarrow$

$C_2 = \frac{25}{3} - 6 = 7/3$

$\therefore \underline{y(x) = 2e^{3x} + \frac{7}{3}xe^{3x}}$

$\boxed{10}$ Char. eq. is $r^3 + 3r^2 - 4r - 12 = 0 \Rightarrow$

$r^2(r+3) - 4(r+3) = 0 \Rightarrow (r^2 - 4)(r+3) = 0 \Rightarrow$

$r = \{-3, -2, 2\}$

$\therefore \underline{y(x) = C_1e^{-3x} + C_2e^{-2x} + C_3e^{2x}}$

$\boxed{\text{I}}$ We wind up with $x(t) = C\ln t + E$

Let t_0 be 10 AM, so $x(t_0) = 0$, $x(t_0+2) = 3$, $x(t_0+4) = 4$

This system results:

$$\begin{cases} C\ln t_0 + E = 0 \\ C\ln(t_0+2) + E = 3 \\ C\ln(t_0+4) + E = 4 \end{cases} \Rightarrow \begin{cases} t_0 = e^{-E/c} & , (1) \\ t_0 + 2 = e^{-E/c} e^{3/c} & , (2) \\ t_0 + 4 = e^{-E/c} e^{4/c} & , (3) \end{cases}$$

Putting (1) into (2) & (3) yields $\begin{cases} t_0 + 2 = t_0 e^{3/c} \Rightarrow \frac{t_0+2}{t_0} = (e^{1/c})^3 \Rightarrow e^{1/c} = \sqrt[3]{\frac{t_0+2}{t_0}} & , (4) \\ t_0 + 4 = t_0 e^{4/c} & , (5) \end{cases}$

Putting (4) into (5) yields $t_0 + 4 = t_0\left(\sqrt[3]{\frac{t_0+2}{t_0}}\right)^4 \Rightarrow \frac{t_0+4}{t_0} = \left(\frac{t_0+2}{t_0}\right)^{4/3} \Rightarrow$

$\left(\frac{t_0+4}{t_0}\right)^3 = \left(\frac{t_0+2}{t_0}\right)^4 \Rightarrow t_0(t_0+4)^3 = (t_0+2)^4 \Rightarrow$

$t_0^4 + 12t_0^3 + 48t_0^2 + 64t_0 = t_0^4 + 8t_0^3 + 24t_0^2 + 32t_0 + 16 \Rightarrow$

$4t_0^3 + 24t_0^2 + 32t_0 - 16 = 0 \Rightarrow$

$t_0^3 + 6t_0^2 + 8t_0 - 4 = 0$

A graphing calculator will yield the one real solution $t_0 \approx 0.3830$ hour $\approx$ 23 minutes

So it began snowing 23 min. before 10:00 AM, or <u>9:37 AM</u>