MATH 250 EXAM #1 KEY (SPRING 2003)

1a Ordinary, 3rd order, independent variable x, dependent variable y, linear

1b Ordinary, 4th order, independent variable x, dependent variable y, nonlinear

1c Ordinary, 1st order, independent variable t, dependent variable r, nonlinear

1d Partial, 2nd order, independent variables r, θ, dependent variable u

2 Let $y = \varphi(x)$ in the equation $\frac{d^2y}{dx^2} + y = 0$:

$$\frac{d^2}{dx^2}(c_1 \sin x + c_2 \cos x) + (c_1 \sin x + c_2 \cos x) = 0 \Rightarrow$$

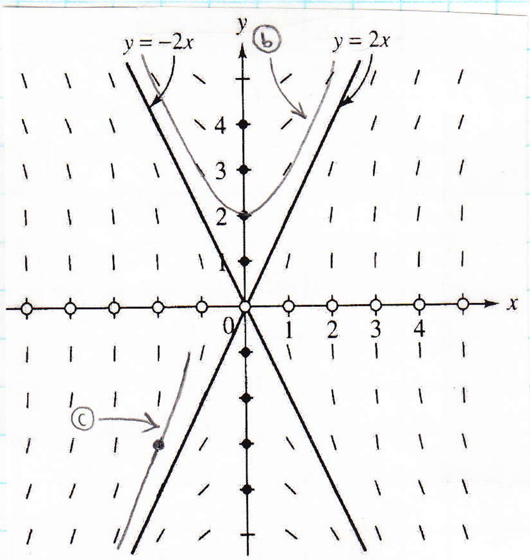
$$-c_1 \sin x - c_2 \cos x + c_1 \sin x + c_2 \cos x = 0 \Rightarrow$$

$(c_1 - c_1)\sin x + (c_2 - c_2)\cos x = 0$, which is true for all values of c_1 & c_2.

3a $\frac{dy}{dx} = \frac{4x}{y} \Rightarrow \frac{d}{dx}(\pm 2x) = \frac{4x}{\pm 2x} \Rightarrow$

$\pm 2 = \pm 2$ ✓

3b & 3c



3d For the solution in (b), we see $y \to 2x$ asymptotically as $x \to \infty$, and $y \to -2x$ asymptotically as $x \to -\infty$

For the solution in (c), we see $y \to 2x$ asymptotically as $x \to -\infty$, whereas the solution curve does not exist as $x \to \infty$.

4 Let $y = \varphi(x) = x^m$, so $\frac{dy}{dx} = mx^{m-1}$ and $\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$. Then equation becomes...

$$3x^2 \cdot m(m-1)x^{m-2} + 11x \cdot mx^{m-1} - 3x^m = 0 \Rightarrow$$

$$3m(m-1)x^m + 11mx^m - 3x^m = 0 \Rightarrow$$

$$[3m(m-1) + 11m - 3]x^m = 0 \Rightarrow$$

$$(3m^2 + 8m - 3)x^m = 0 \Rightarrow$$

$$3m^2 + 8m - 3 = 0 \Rightarrow (3m-1)(m+3) = 0 \Rightarrow$$

$$m = \{1/3, -3\}$$

$\therefore \varphi(x) = \{\sqrt[3]{x}, 1/x^3\}$ are solutions.

5 $y' = \frac{y^2 + y}{2x} \Rightarrow f(x_n, y_n) = \frac{y_n^2 + y_n}{2x_n}$

$y(1) = 1 \Rightarrow x_0 = 1$ & $y_0 = 1$

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.2f(1,1) = 1 + 0.2\left[\frac{1^2+1}{2(1)}\right] = 1.2$$

$$y_2 = y_1 + hf(x_1, y_1) = 1.2 + 0.2f(1.2, 1.2) = 1.2 + 0.2\left[\frac{1.2^2 + 1.2}{2(1.2)}\right] = 1.42$$

$$y_3 = y_2 + hf(x_2, y_2) = 1.42 + 0.2f(1.4, 1.42) = 1.42 + 0.2\left[\frac{1.42^2 + 1.42}{2(1.4)}\right] = 1.665457$$

$$y_4 = 1.665457 + 0.2f(1.6, 1.665457) = 1.942907$$

So $(x_1, y_1) = (1.2, 1.2)$, $(x_2, y_2) = (1.4, 1.42)$, $(x_3, y_3) = (1.6, 1.665457)$, $(x_4, y_4) = (1.8, 1.942907)$

6 $\frac{dy}{2\sqrt{y+1}} = \cos x\, dx \Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{y+1}}dy = \int \cos x\, dx$

$\frac{1}{2} \cdot 2(y+1)^{1/2} = \sin x + C \Rightarrow \sqrt{y+1} = \sin x + C$

Since $y(\pi) = 0$, $\sqrt{0+1} = \sin \pi + C \Rightarrow C = 1$

$\therefore$ $\underline{\sqrt{y+1} = \sin x + 1}$ is the solution

Or: $y + 1 = (\sin x + 1)^2 = \sin^2 x + 2\sin x + 1 \Rightarrow$

$\underline{y = \sin^2 x + 2\sin x}$ is the (explicit) solution.

7 $150\frac{dv}{dt} = 150(9.8) - 5v$, $v(0) = 12$ m/s $\Rightarrow \frac{dv}{dt} = 9.8 - \frac{v}{30} \Rightarrow \frac{dv}{dt} = \frac{294 - v}{30} \Rightarrow$

$\frac{30}{294 - v}dv = dt \Rightarrow -30\int \frac{1}{v - 294}dv = t + C_1 \Rightarrow -30\ln|v - 294| = t + C_1 \Rightarrow$

$|v - 294| = e^{-t/30 + C_2} = C_3 e^{-t/30}$, $C_3 > 0$

So $v - 294 = \pm C_3 e^{-t/30} \Rightarrow v = 294 + Ke^{-t/30}$, $K = \pm C_3 \neq 0$

Since $v(0) = 12$, $12 = 294 + Ke^0 \Rightarrow K = -282$

$\therefore v(t) = 294 - 282e^{-t/30}$

$\lim_{t\to\infty} v(t) = \underline{294 \text{ m/s}}$

8a In standard form: $\frac{dy}{dx} + 4y = x^2 e^{-4x}$

So $P(x) = 4$ & $Q(x) = x^2 e^{-4x}$

Then $\mu(x) = e^{\int P(x)dx} = e^{\int 4dx} = e^{4x}$

So $y(x) = \frac{1}{e^{4x}}\int e^{4x}\cdot x^2 e^{-4x}dx = e^{-4x}\int x^2 dx = e^{-4x}\left(\frac{1}{3}x^3 + C\right) = \underline{\frac{1}{3}x^3 e^{-4x} + Ce^{-4x}}$

8b Standard form: $\frac{dy}{dx} + y\cot x = x$, $y(\pi/2) = 2$

So $P(x) = \cot x$ & $Q(x) = x$

Then $\mu(x) = e^{\int \cot x} = e^{\ln|\sin x|} = |\sin x|$

So $y(x) = \frac{1}{|\sin x|}\int x\sin x\, dx = \frac{1}{|\sin x|}(\sin x - x\cos x + C)$

Since $y(\pi/2) = 2$, $2 = \frac{1}{\sin(\pi/2)}\left[\sin(\pi/2) - \pi/2\cos(\pi/2) + C\right] \Rightarrow$

$2 = 1\cdot(1 - 0 + C) = 1 + C \Rightarrow C = 1$

$\therefore y(x) = \frac{1}{\sin x}(\sin x - x\cos x + 1) = \underline{1 - x\cot x + \csc x}$

(Note: $|\sin x|$ becomes $\sin x$ since $x = \pi/2$ in the initial condition, and $\sin(\pi/2) > 0$)