Math 250 Exam #5 Key (Fall 2021)

1 Let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. Putting this into the ODE, along with the Frobenius series for y' and y'', gives

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + 5\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + x\sum_{n=0}^{\infty} c_n x^{n+r} = 0,$$

so that

$$(2r^2+3r)c_0x^{r-1} + (r+1)(2r+5)c_1x^r + \sum_{n=2}^{\infty}\big[2(n+r)(n+r-1)c_n + 5(n+r)c_n + c_{n-2}\big]x^{n+r-1} = 0.$$

The coefficients of all powers of x must equal zero. Since the indicial equation $2r^2+3r=0$ implies $r = 0, -\frac{3}{2}$ (and leaves c_0 to be arbitrary), we can have $(r+1)(2r+5)c_1 = 0$ only if $c_1 = 0$. For $n \geq 2$ we have

$$2(n+r)(n+r-1)c_n + 5(n+r)c_n + c_{n-2} = 0,$$

and thus

$$c_n = -\frac{c_{n-2}}{(n+r)(2n+2r+3)}, \quad n \geq 2.$$

For the index value $r = 0$ the recurrence relation is

$$c_n = -\frac{c_{n-2}}{n(2n+3)}, \quad n \geq 2.$$

Because $c_1 = 0$, we find that $c_{2k+1} = 0$ for all $k \geq 1$. Meanwhile,

$$c_2 = -\frac{c_0}{2\cdot 7}, \quad c_4 = \frac{c_0}{(2\cdot 4)(7\cdot 11)}, \quad c_6 = -\frac{c_0}{(2\cdot 4\cdot 6)(7\cdot 11\cdot 15)},$$

and so on. If we let $c_0 = 1$, we get a series solution y_1 of the form

$$\begin{aligned} y_1 &= 1 - \frac{x^2}{2\cdot 7} + \frac{x^4}{(2\cdot 4)(7\cdot 11)} - \frac{x^6}{(2\cdot 4\cdot 6)(7\cdot 11\cdot 15)} + \cdots \\ &= 1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2\cdot 4\cdot 6\cdots 2n)[7\cdot 11\cdot 15\cdots(4n+3)]}. \end{aligned}$$

For the index value $r = -\frac{3}{2}$ the recurrence relation is

$$c_n = -\frac{c_{n-2}}{n(2n-3)}, \quad n \geq 2.$$

Again $c_{2k+1} = 0$ for all $k \geq 1$, while

$$c_2 = -\frac{c_0}{1\cdot 2}, \quad c_4 = \frac{c_0}{(2\cdot 4)(1\cdot 5)}, \quad c_6 = -\frac{c_0}{(2\cdot 4\cdot 6)(1\cdot 5\cdot 9)},$$

and so on. If we let $c_0 = 1$, we get a series solution y_2 of the form

$$\begin{aligned} y_2 &= x^{-3/2} - \frac{x^{1/2}}{2\cdot 1} + \frac{x^{5/2}}{(2\cdot 4)(1\cdot 5)} - \frac{x^{9/2}}{(2\cdot 4\cdot 6)(1\cdot 5\cdot 9)} + \cdots \\ &= x^{-3/2} + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-3/2}}{(2\cdot 4\cdot 6\cdots 2n)[1\cdot 5\cdot 9\cdots(4n-3)]}. \end{aligned}$$

The general solution to the ODE is

$$y = d_1 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2 \cdot 4 \cdot 6 \cdots 2n)[7 \cdot 11 \cdot 15 \cdots (4n+3)]} \right] + d_2 x^{-3/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2 \cdot 4 \cdot 6 \cdots 2n)[1 \cdot 5 \cdot 9 \cdots (4n-3)]} \right]$$

for arbitrary d_1, d_2.

2 Write as

$$y'' + 4y = t - \left(t - \tfrac{\pi}{2}\right) u\left(t - \tfrac{\pi}{2}\right).$$

Taking the Laplace transform yields

$$(s^2+4)Y = \frac{1}{s^2} - \frac{e^{-(\pi/2)s}}{s^2} \quad \hookrightarrow \quad Y = \frac{1}{s^2(s^2+4)} - \frac{e^{-(\pi/2)s}}{s^2(s^2+4)}.$$

Then

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] - \mathcal{L}^{-1}\left[\frac{e^{-(\pi/2)s}}{s^2(s^2+4)}\right],$$

and finally

$$y(t) = \frac{t}{4} - \frac{\sin 2t}{8} - \left[\frac{1}{4}\left(t - \frac{\pi}{2}\right) - \frac{1}{8}\sin(2t - \pi)\right] u\left(t - \frac{\pi}{2}\right).$$

3 The Laplace transform of the equation is

$$sY - 1 = \frac{s}{s^2+1} + \mathcal{L}[\cos t]\mathcal{L}[y(t)] = \frac{s}{s^2+1} + \frac{s}{s^2+1}Y.$$

Solving for Y:

$$Y = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}.$$

Finally,

$$y(t) = \frac{1}{2}t^3 + t + 1.$$

4 The Laplace transform of the equation is

$$s^2Y - 2 + 2sY + 2Y = e^{-\pi s} \quad \hookrightarrow \quad Y = \frac{2}{(s+1)^2+1} + e^{-\pi s} \cdot \frac{2}{(s+1)^2+1},$$

and so

$$y(t) = 2\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right] + 2\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{(s+1)^2+1}\right],$$

giving

$$y(t) = 2e^{-t}\sin t + 2e^{-(t-\pi)}\sin(t-\pi)u(t-\pi).$$