Math 250 Exam #4 Key (Fall 2021)

1 Model: $50x'' + 200x = 0$, $x(0) = 0$, $x'(0) = -10$. Here $x(t) < 0$ is the position that compresses the (vertically hanging) spring. From the ODE comes

$$x(t) = c_1 \cos 2t + c_2 \sin 2t,$$

and with the initial conditions we find $x(t) = -5\sin 2t$. Period of motion is π seconds. Now find t such that $x'(t) = 5$ m/s, or $\cos 2t = -\frac{1}{2}$. The times are

$$t = \frac{\pi}{3} + \pi n \quad \text{and} \quad t = \frac{2\pi}{3} + \pi n,$$

$n \geq 0$ an integer.

2 Put $y = \sum_{n=0}^{\infty} c_n x^n$ into the ODE to get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0,$$

so

$$\sum_{n=0}^{\infty} \big[(n+2)(n+1)c_{n+2} + nc_n + 2c_n\big]x^n = 0,$$

and hence $(n+2)(n+1)c_{n+2} + nc_n + 2c_n = 0$ for all $n \geq 0$. Solve for c_{n+2}:

$$c_{n+2} = -\frac{c_n}{n+1}, \qquad n \geq 0.$$

We use this recurrence relation to find

$$c_2 = -c_0,\ \ c_4 = \frac{c_0}{1\cdot 3},\ \ c_6 = -\frac{c_0}{1\cdot 3\cdot 5},\ \ c_8 = \frac{c_0}{1\cdot 3\cdot 5\cdot 7}, \cdots$$

and generally

$$c_{2n} = \frac{(-1)^n c_0}{(1)(3)(5)\cdots(2n-1)}, \qquad n \geq 1.$$

Also we find

$$c_3 = -\frac{c_1}{2},\ \ c_5 = \frac{c_1}{2\cdot 4},\ \ c_7 = -\frac{c_1}{2\cdot 4\cdot 6}, \cdots$$

and generally

$$c_{2n+1} = \frac{(-1)^n c_1}{(2)(4)(6)\cdots(2n)}, \qquad n \geq 1.$$

Since c_0 and c_1 are left arbitrary, setting $c_0 = 0$ and $c_1 = 1$ results in $y = \sum_{n=0}^{\infty} c_n x^n$ becoming

$$y_1(x) = \sum_{n=0}^{\infty} c_{2n+1}x^{2n+1} = x + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2)(4)(6)\cdots(2n)}x^{2n+1}.$$

Letting $c_0 = 1$ and $c_1 = 0$ gives

$$y_2(x) = \sum_{n=0}^{\infty} c_{2n}x^{2n} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(1)(3)(5)\cdots(2n-1)}x^{2n}.$$

The general solution (not asked for here) is $y = d_1y_1 + d_2y_2$ for arbitrary d_1, d_2.

3 Laplace transform of ODE is $(sY+3)+3Y=2/s$, so that

$$Y(s)=\frac{2-3s}{s(s+3)}=\frac{2/3}{s}-\frac{11/3}{s+3},$$

and then

$$y(t)=\mathcal{L}^{-1}[Y]=\frac{2}{3}-\frac{11}{3}e^{-3t}.$$

4 Going through the usual grind,

$$y(t)=5te^{t}+\frac{1}{2}t^{2}e^{t}.$$