Math 250 Exam #3 Key (Fall 2021)

1 The complementary solution to the ODE is $y_c = c_1 + c_2 e^{-x}$. Now we solve $y'' + y' = x$. A particular solution has the form $Ax^2 + Bx$, and the Method of Undetermined Coefficients reveals that $A = \frac{1}{2}$ and $B = -1$. Thus we get $y_{p_1} = \frac{1}{2}x^2 - x$.

Next solve $y'' + y' = \sin 2x$. A particular solution has the form $A\cos 2x + B\sin 2x$, and once it's found that $A = -\frac{1}{10}$ and $B = -\frac{1}{5}$, we get $y_{p_2} = -\frac{1}{10}\cos 2x - \frac{1}{5}\sin 2x$.

By the Superposition Principle the general solution to $y'' + y' = x + \sin 2x$ is

$$\boldsymbol{y = y_c + y_{p_1} + y_{p_2} = c_1 + c_2 e^{-x} + \tfrac{1}{2}x^2 - x - \tfrac{1}{10}\cos 2x - \tfrac{1}{5}\sin 2x.}$$

2 Auxiliary equation $r^2 + 1 = 0$ has roots $r = \pm i$, so complementary solution is $y_c = c_1\cos x + c_2\sin x$. From this we obtain two linearly independent solutions to the ODE: $y_1 = \cos x$ and $y_2 = \sin x$. The Wronskian is $W[y_1, y_2](x) \equiv 1$. We now find

$$u_1(x) = -\int \sin x \sec^3 x\,dx = -\frac{1}{2}\sec^2 x \quad \text{and} \quad u_2(x) = \int \cos x \sec^3 x\,dx = \tan x,$$

so a particular solution is

$$y_p = u_1(x)y_1 + u_2(x)y_2 = -\tfrac{1}{2}\sec x + \sin^2 x \sec x = \tfrac{1}{2}\sec x - \cos x.$$

General solution is

$$\boldsymbol{y = y_c + y_p = c_1\cos x + c_2\sin x + \tfrac{1}{2}\sec x - \cos x.}$$

(Note: different outcomes for $u_1(x)$ are possible depending on what constant term is added. We could write, say, $u_1(x) = -\frac{1}{2}\sec^2 x + \frac{1}{2} = -\frac{1}{2}\tan^2 x$.)

3 Apply, say, $D + 1$ to the 2nd equation in the system and multiply the 1st equation by 3 to get

$$\begin{cases} 3(D+1)x + \qquad\quad 3(D-1)y = 6 \\ 3(D+1)x + (D+1)(D+2)y = -1 \end{cases}$$

Subtract to get

$$3(D-1)y - (D+1)(D+2)y = 7 \quad \hookrightarrow \quad y'' + 5y = -7.$$

The auxiliary equation $r^2 + 5 = 0$ has roots $r = \pm i\sqrt{5}$, so the complementary solution is

$$y_c = c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t.$$

Almost by inspection we have $y_p = -\frac{7}{5}$ as a particular solution to $y'' + 5y = -7$. General solution:

$$y = c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t - \tfrac{7}{5}. \tag{1}$$

Next apply $D+2$ and $D-1$ for the 1st and 2nd equation in the system, respectively, giving

$$\begin{cases} (D+2)(D+1)x + (D+2)(D-1)y = 4 \\ \qquad\quad 3(D-1)x + (D-1)(D+2)y = 1 \end{cases}$$

Subtract to get

$$(D+2)(D+1)x - 3(D-1)x = 3 \quad \hookrightarrow \quad (D^2+5)x = 3 \quad \hookrightarrow \quad x'' + 5x = 3.$$

By similar means as before, we get the general solution

$$x = c_3 \cos \sqrt{5}t + c_4 \sin \sqrt{5}t + \tfrac{3}{5}. \qquad (2)$$

Now put (1) and (2) into, say, the 2nd equation in the original system to get, after some rearrangement of terms,

$$(3c_3 + \sqrt{5}c_2 + 2c_1)\cos \sqrt{5}t + (3c_4 + 2c_2 - \sqrt{5}c_1)\sin \sqrt{5}t = 0.$$

Hence $c_3 = -\frac{2}{3}c_1 - \frac{\sqrt{5}}{3}c_2$ and $c_4 = \frac{\sqrt{5}}{3}c_1 - \frac{2}{3}c_2$. The general solution to the system is:

$$\boldsymbol{x(t) = (-\tfrac{2}{3}c_1 - \tfrac{\sqrt{5}}{3}c_2)\cos \sqrt{5}t + (\tfrac{\sqrt{5}}{3}c_1 - \tfrac{2}{3}c_2)\sin \sqrt{5}t + \tfrac{3}{5},}$$
$$\boldsymbol{y(t) = c_1 \cos \sqrt{5}t + c_2 \sin \sqrt{5}t - \tfrac{7}{5}.}$$

4 Let $u = y'$, so $u' = y''$ and the ODE becomes $x^2u' + u^2 = 0$, which is separable:

$$-\int \frac{1}{u^2}\,du = \int \frac{1}{x^2}\,dx \quad \hookrightarrow \quad \frac{1}{u} = -\frac{1}{x} + c_1 = \frac{c_1x - 1}{x} \quad \hookrightarrow \quad \frac{dy}{dx} = \frac{x}{c_1x - 1}.$$

If $c_1 = 0$ we get

$$\frac{dy}{dx} = -x \quad \hookrightarrow \quad \int dy = -\int x\,dx \quad \hookrightarrow \quad \boldsymbol{y = -\frac{x^2}{2} + c,}$$

a one-parameter family of solutions to the original ODE. If $c_1 \neq 0$ we get

$$\frac{dy}{dx} = \frac{x}{c_1x - 1} = \frac{1}{c_1} + \frac{1/c_1}{c_1x - 1} \quad \hookrightarrow \quad \int dy = \int \left(\frac{1}{c_1} + \frac{1/c_1}{c_1x - 1}\right) dx,$$

which leads to a two-parameter family of solutions:

$$\boldsymbol{y = \frac{x}{c_1} + \frac{\ln|c_1x - 1|}{c_1^2} + c_2,}$$

This family does not include the members of the one-parameter family above.

5 In general we have

$$y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y(4)(0)}{4!}x^4 + \cdots.$$

Now,

$$y'' = 4x + 2y^2 \quad \Rightarrow \quad y''(0) = 2y^2(0) = 2(-1)^2 = 2,$$

and

$$y''' = 4 + 4yy' \quad \Rightarrow \quad y'''(0) = 4 + 4y(0)y'(0) = 4 + 4(-1)(2) = -4,$$

Therefore

$$\boldsymbol{y(x) \approx -1 + 2x + x^2 - \frac{2}{3}x^3.}$$