Math 250 Exam #2 Key (Fall 2021)

1 Write the DE as

$$y' = -\frac{1+(y/x)^2}{2(y/x)},$$

then let $u = y/x$ so that $y = xu$ and $y' = u + xu'$, and the DE becomes

$$u + xu' = -\frac{1+u^2}{2u}.$$

The equation is separable, giving

$$-\int \frac{2u}{1+3u^2}\,du = \int \frac{1}{x}\,dx \quad \Rightarrow \quad -\frac{1}{3}\ln|3u^2+1| = \ln|x| + c \quad \Rightarrow \quad \ln\left(\frac{3y^2}{x^2}+1\right) = \ln|x|^{-3} + c.$$

(Note the elimination of absolute values around the expression that cannot be negative.) From this comes

$$\frac{3y^2}{x^2} + 1 = \hat{c}|x|^{-3} = \pm\hat{c}x^{-3}, \quad \hat{c} > 0,$$

and then

$$\frac{3y^2}{x^2} + 1 = c_0 x^{-3}, \quad c_0 \neq 0.$$

The solution may also be written as $x(3y^2 + x^2) = c_0$, which makes clear that in this case we cannot allow $c_0 = 0$, for then we would require $3y^2 + x^2 = 0$ for x on some interval of real numbers, which is impossible!

2 Let $u = 2x + y + 1$, so that $y' = u' - 2$ and the DE becomes $u(u' - 2) = 1$. This is separable, giving

$$\int \frac{u}{2u+1}\,du = \int dx \quad \Rightarrow \quad \frac{u}{2} - \frac{\ln|2u+1|}{4} = x + c.$$

Since $u = 2x + y + 1$, we next obtain

$$\frac{2x+y+1}{2} - \frac{\ln|4x+2y+3|}{4} = x + c \quad \Rightarrow \quad 2y + 2 - \ln|4x+2y+3| = c.$$

The constant term 2 can be absorbed by c, so that

$$2y - \ln|4x+2y+3| = c \quad \Rightarrow \quad 4x + 2y + 3 = c_0 e^{2y}, \quad c_0 \neq 0$$

If we let $c_0 = 0$ we obtain $y = -2x - \frac{3}{2}$, which by direct substitution is found to satisfy the original DE. Thus we write the solution as

$$4x + 2y + 3 = ce^{2y},$$

where $c \in (-\infty, \infty)$ as before.

3 Newton's Law of Cooling states that $T'(t) = k[T(t) - T_a]$. Here we have $T_a = -5$, $T(1) = 10$, and $T(4) = 0$. Now, noting that $T(t) > -5$ for all $t \geq 0$,

$$T' = k(T+5) \quad \Rightarrow \quad \int \frac{1}{T+5}dT = \int k\,dt \quad \Rightarrow \quad \ln|T+5| = kt + c \quad \Rightarrow \quad T + 5 = e^{kt+c},$$

and so

$$T(t) = -5 + Ce^{kt}.$$

From $T(1) = 10$ we obtain

$$10 = -5 + Ce^k \quad \Rightarrow \quad Ce^k = 15 \quad \Rightarrow \quad C = 15e^{-k},$$

and so

$$T(t) = -5 + 15e^{-k}e^{kt} = -5 + 15e^{k(t-1)}.$$

From $T(4) = 0$ we obtain

$$0 = -5 + 15e^{3k} \quad \Rightarrow \quad e^{3k} = \frac{1}{3} \quad \Rightarrow \quad 3k = \ln\left(\frac{1}{3}\right) \quad \Rightarrow \quad k = \frac{1}{3}\ln\left(\frac{1}{3}\right) \approx -0.366.$$

Thus

$$T(t) = -5 + 15e^{\frac{t-1}{3}\ln\frac{1}{3}} = -5 + 15\left(\frac{1}{3}\right)^{(t-1)/3} = -5 + 15\sqrt[3]{\frac{1}{3^{t-1}}}.$$

The temperature in the kitchen is

$$T(0) = -5 + 15\sqrt[3]{3} \approx 62^\circ \text{ C}.$$

4 Let $x(t)$ be the mass of salt, in kilograms, in the tank at time t, so that $x(0) = 30$. The volume of solution in the tank is $V(t) = 200 + 2t$. The full derivation of $x'(t)$, which is the rate of change of the amount of salt in the tank at time t, is as follows:

$$\begin{aligned} x'(t) &= (\text{rate salt enters Tank 1}) - (\text{rate salt leaves Tank 1}) \\ &= \left(\frac{0.3 \text{ kg}}{1 \text{ L}}\right)\left(\frac{4 \text{ L}}{1 \text{ min}}\right) - \left(\frac{x(t) \text{ kg}}{V(t) \text{ L}}\right)\left(\frac{2 \text{ L}}{1 \text{ min}}\right) \\ &= 1.2 - \frac{2x(t)}{200 + 2t}. \end{aligned}$$

Thus we have a linear first-order ODE:

$$x' + \frac{x}{t + 100} = \frac{6}{5}.$$

To solve this equation, we multiply by the integrating factor

$$\mu(t) = \exp\left(\int \frac{1}{t + 100}\,dt\right) = e^{\ln(t+100)} = t + 100$$

to obtain

$$(t + 100)x' + x = \frac{6}{5}(t + 100),$$

which becomes

$$[(t + 100)x]' = \frac{6}{5}(t + 100)$$

and thus

$$(t + 100)x = \frac{6}{5}\int (t + 100)\,dt = \frac{3}{5}t^2 + 120t + c.$$

From this we get a general explicit solution to the ODE,

$$x(t) = \frac{3t^2 + 600t + c}{5t + 500}.$$

To determine c we use the initial condition $x(0) = 30$, giving $c/500 = 30$, and thus $c = 15{,}000$. So, the amount of salt in the tank at time t is given by

$$x(t) = \frac{3t^2 + 600t + 15,000}{5t + 500}.$$

The tank is full when $t = 150$ minutes. At that time the concentration of salt is:

$$\frac{x(150)}{V(150)} = \frac{1}{500}\left[\frac{3(150)^2 + 600(150) + 15,000}{5(150) + 500}\right] = \frac{138}{500} = 0.276 \text{ kg/L}.$$

5 Write the equation in the form

$$y'' - \frac{2x}{1-x^2}y' + \frac{2}{1-x^2} = 0,$$

so $P(x) = -2x/(1-x^2)$. Now,

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\,dx = x\int\left(\frac{1}{x^2}\exp\left(\int \frac{2x}{1-x^2}\,dx\right)\right)dx, \tag{1}$$

where

$$\int \frac{2x}{1-x^2}\,dx = -\ln|1-x^2| + c = -\ln(1-x^2) + c,$$

the absolute values disappearing since we assume $x \in (-1, 1)$. Putting this into (1) with $c = 0$ results in

$$\begin{aligned} y_2(x) &= x\int \frac{1}{x^2(1-x^2)}\,dx = x\int\left(\frac{1}{x^2} + \frac{1/2}{1-x} + \frac{1/2}{1+x}\right)dx \\ &= x\left(-\frac{1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) + cx\right) \end{aligned}$$

for $-1 < x < 1$. Letting $c = 0$ again, we get

$$y_2(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1.$$

6 With auxiliary equation $6r^2 - 11r + 4 = 0$, which has roots $r = \frac{4}{3}, \frac{1}{2}$, we get

$$y = c_1e^{4x/3} + c_2e^{x/2}$$

as the general solution to the DE. For the IVP the particular solution is

$$y = -3e^{4x/3} + 4e^{x/2}.$$

7 With a bit of trial-and-error we find the auxiliary equation $3r^3 + 5r^2 + r - 1 = 0$ has root $r = -1$, and so factors as

$$(r+1)(3r^2 + 2r - 1) = 0 \;\;\Rightarrow\;\; (r+1)^2(3r-1) = 0.$$

Thus $\frac{1}{3}$ is another root and -1 is a double root. General solution:

$$y = c_1e^{-x} + c_2xe^{-x} + c_3e^{x/3}.$$