Math 250 Exam #1 Key (Fall 2021)

1 On $(-\infty, 0)$ and $(0, \infty)$ the function $y(x)$ equals a polynomial function, and so is differentiable on those intervals. Using rules of differentiation we have $y'(x) = -2x$ for $x < 0$ and $y'(x) = 2x$ for $x > 0$. It is not a given that $y(x)$ is differentiable at $x = 0$, however, and so rules of differentiation cannot be assumed to apply. Using the definition of derivative,

$$y'(0) = \lim_{h\to 0} \frac{y(h) - y(0)}{h} = \lim_{h\to 0} \frac{y(h)}{h},$$

but to determine this limit we'll need to work with one-sided limits: since

$$\lim_{h\to 0^+} \frac{y(h)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0$$

and

$$\lim_{h\to 0^-} \frac{y(h)}{h} = \lim_{h\to 0^-} \frac{-h^2}{h} = \lim_{h\to 0^-} (-h) = 0,$$

we have $y'(0) = 0$.

Now, for $x < 0$ we have $y(x) = -x^2$ and $y'(x) = -2x$. Putting this into $xy' - 2y = 0$ yields $0 = 0$. For $x > 0$ we have $y(x) = x^2$ and $y'(x) = x^2$, which again results in $0 = 0$ in the DE. Finally, $y(0) = y'(0) = 0$, which again satisfies the DE. Thus y satisfies the DE for all $x \in \mathbb{R}$

2 Here $y' = f(x, y)$ with

$$f(x, y) = \frac{\sqrt{y}}{3x^2 - 4y}$$

and

$$f_y(x, y) = \frac{3x^2 + 4y}{(3x^2 - 4y)^2}.$$

The domains of f and f_y are

$$\operatorname{Dom} f = \{(x, y) : y \geq 0 \text{ and } y \neq \tfrac{3}{4}x^2\}$$

and

$$\operatorname{Dom} f_y = \{(x, y) : y > 0 \text{ and } y \neq \tfrac{3}{4}x^2\}.$$

Both functions are continuous on their domains, and since the intersection of these domains equals $\operatorname{Dom} f_y$ itself, we are assured of our IVP having a unique solution provided that the point (x_0, y_0) is such that $y_0 > 0$ and $y_0 \neq \frac{3}{4}x_0^2$.

3 The equation is separable, giving

$$\int y\, dy = \int (1 - 2x)\, dx \quad \Rightarrow \quad y^2 = 2x - 2x^2 + c.$$

This means $y = \pm\sqrt{2x - 2x^2 + c}$, but since $y(1) = -2 < 0$ we must have $y = -\sqrt{2x - 2x^2 + c}$ (square roots are never negative). Now, $y(1) = -2$ implies $-2 = -\sqrt{2(1) - 2(1)^2 + c}$, so that $c = 4$ and the solution to the IVP is

$$y = -\sqrt{4 + 2x - 2x^2}.$$

This solution is valid on its domain, since the original DE itself rules out no values of x. That is, the interval of validity I is the interval for x on which $4 + 2x - 2x^2 \geq 0$, or equivalently $x^2 - x - 2 \leq 0$, and thus $(x - 2)(x + 1) \leq 0$. Solving this inequality, we find $I = [-1, 2]$.

4 Equation is separable:

$$\int \frac{1}{y+1}\,dy = \int x^2\,dx \quad \Rightarrow \quad \ln|y+1| = \tfrac{1}{3}x^3 + c \quad \Rightarrow \quad |y+1| = e^c e^{x^3/3} = Ce^{x^3/3},$$

where $C = e^c > 0$ is arbitrary. Then $y + 1 = \pm Ce^{x^3/3} = C_0 e^{x^3/3}$ for $C_0 \neq 0$. If we let $C_0 = 0$ we get $y \equiv -1$ (i.e. y is constantly equal to -1), and it's easy to check that this satisfies the DE. We may thus let C_0 be any real number, and so revert to denoting the arbitrary constant by c. Solution is $y = ce^{x^3/3} - 1$, $c \in \mathbb{R}$. Interval of validity is $(-\infty, \infty)$.

5 In the standard (or normal) form the DE is

$$y' - \frac{1}{x}y = 2x + 1,$$

which immediately makes clear that $x \neq 0$. Thus no solution curve passes through the y-axis, which breaks the family of solutions into two "subfamilies": one on the left half-plane, and another on the right half-plane. Since the initial condition $y(-1) = 8$ requires a solution that passes through the point $(-1, 8)$, we only need focus on those solutions to the DE whose graphs are on the left half-plane where $x < 0$. This means $|x| = -x$, a fact we use shortly.

The DE is linear, so we get an integrating factor:

$$\mu(x) = \exp\left(\int -\frac{1}{x}\,dx\right) = e^{-\ln|x|} = \frac{1}{|x|} = -\frac{1}{x}.$$

Multiplying the DE by $-1/x$ gives

$$-\frac{y'}{x} + \frac{y}{x^2} = -2 - \frac{1}{x} \quad \Rightarrow \quad (-y/x)' = -2 - \frac{1}{x} \quad \Rightarrow \quad -\frac{y}{x} = -2x - \ln|x| + c,$$

and then $y = 2x^2 + x\ln(-x) - cx$. Using $y(-1) = 8$ in this last equation leads to $c = 6$, and therefore the solution to the IVP is

$$y = 2x^2 + x\ln(-x) - 6x$$

with interval of validity $(-\infty, 0)$.

6 The equation is given to be exact, with

$$M(x, y) = ye^{xy} - \frac{1}{y}$$

and

$$N(x, y) = xe^{xy} + \frac{x}{y^2}.$$

We find a function F such that $F_x = M$ and $F_y = N$. Now,

$$F(x, y) = \int F_x(x, y)\,dx = \int M(x, y)\,dx = \int \left(ye^{xy} - \frac{1}{y}\right)dx = e^{xy} - \frac{x}{y} + g(y).$$

Differentiating with respect to y then yields

$$F_y(x, y) = xe^{xy} + \frac{x}{y^2} + g'(y),$$

where $F_y = N$ implies that

$$xe^{xy} + \frac{x}{y^2} + g'(y) = xe^{xy} + \frac{x}{y^2},$$

so that $g'(y) = 0$ and hence $g(y) = c_1$ for some arbitrary constant c_1. This leaves us with

$$F(x, y) = e^{xy} - \frac{x}{y} + c_1.$$

The general implicit solution to the ODE is given by $F(x, y) = c_2$ for arbitrary constant c_2, which here becomes

$$e^{xy} - \frac{x}{y} + c_1 = c_2.$$

Letting $c = c_2 - c_1$, we finally write $ye^{xy} - x = cy$.