Math 250 Exam #4 Key (Fall 2020)

1 Let $y = \sum_{n=0}^{\infty} c_n x^n$, so differential equation (DE) is

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 6\sum_{n=0}^{\infty} c_n x^n = 0,$$

which with reindexing becomes

$$(2c_2 - 6c_0) + (6c_3 - 6c_1)x + \sum_{n=2}^{\infty}\big[(n-3)(n+2)c_n + (n+1)(n+2)c_{n+2}\big]x^n = 0.$$

Hence $c_2 = 3c_0$, $c_3 = c_1$, and

$$c_{n+2} = \frac{3-n}{n+1}c_n$$

for $n \geq 2$. With this we find that $c_4 = c_0$, $c_5 = 0$, $c_6 = -\frac{1}{5}c_0$, $c_7 = 0$, $c_8 = \frac{3}{5\cdot 7}c_0$, $c_9 = 0$, $c_{10} = -\frac{3}{7\cdot 9}c_0$, $c_{11} = 0$, $c_{12} = \frac{3}{9\cdot 11}c_0$, etc. So

$$\begin{aligned} y &= c_0 + c_1 x + 3c_0x^2 + c_1x^3 + c_0x^4 - \frac{1}{5}c_0x^6 + \frac{3}{5\cdot 7}c_0x^8 - \frac{3}{7\cdot 9}c_0x^{10} + \cdots \\ &= c_1(x+x^3) + c_0\left(1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \frac{3}{5\cdot 7}x^8 - \frac{3}{7\cdot 9}x^{10} + \cdots\right). \end{aligned}$$

Letting $c_1 = 1$, $c_0 = 0$, and $c_1 = 0$, $c_0 = 1$, we obtain two particular, linearly indepedent solutions to the DE:

$$y_1 = x + x^3, \quad y_2 = \sum_{n=0}^{\infty} \frac{3(-1)^n}{(2n-3)(2n-1)}x^{2n}.$$

2 By definition,

$$\begin{aligned} \mathcal{L}[f](s) &= \int_0^{\infty} e^{-st}f(t)\,dt = \int_0^2 te^{-st}\,dt + \int_2^{\infty}(3-t)e^{-st}\,dt \\ &= \left[\left(-\frac{t}{s} - \frac{1}{s^2}\right)e^{-st}\right]_0^2 + \left[-\frac{3}{s}e^{-st}\right]_2^{\infty} - \left[\left(-\frac{t}{s} - \frac{1}{s^2}\right)e^{-st}\right]_0^{\infty} \\ &= \frac{1}{s^2} - \left(\frac{1}{s} + \frac{2}{s^2}\right)e^{-2s}. \end{aligned}$$

3a Using the table provided,

$$\mathcal{L}^{-1}\left[\frac{15s}{2s^2+50}\right] = \frac{15}{2}\mathcal{L}^{-1}\left[\frac{s}{s^2+5^2}\right] = \frac{15}{2}\cos(5t).$$

3b Using partial fraction decomposition and the table provided,

$$\mathcal{L}^{-1}\left[\frac{s+1}{s^2-4s}\right] = \mathcal{L}^{-1}\left[\frac{-1/4}{s} + \frac{5/4}{s-4}\right] = -\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{5}{4}\mathcal{L}^{-1}\left[\frac{1}{s-4}\right] = -\frac{1}{4} + \frac{5}{4}e^{4t}.$$

4 Letting $Y(s) = \mathcal{L}[y](s)$, from $\mathcal{L}[y'' + 9y] = \mathcal{L}[e^t]$ and the initial conditions we obtain

$$s^2Y - sy(0) - y'(0) + 9Y = \frac{1}{s-4} \quad \Rightarrow \quad Y = \frac{1}{(s-1)(s^2+9)} = \frac{\frac{1}{10}}{s-1} - \frac{\frac{1}{10}s + \frac{1}{10}}{s^2+9},$$

and thus

$$\begin{aligned} y(t) &= \frac{1}{10}\mathcal{L}^{-1}\left[\frac{1}{s-1} - \frac{s+1}{s^2+9}\right] = \frac{1}{10}\left(e^t - \mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2+9}\right]\right) \\ &= \frac{1}{10}\left(e^t - \cos 3t - \frac{1}{3}\sin 3t\right). \end{aligned}$$