Math 250 Exam #3 Key (Fall 2020)

1 Let $u = \dfrac{dy}{dx}$, so that $u' = \dfrac{du}{dx} = \dfrac{du}{dy}\dfrac{dy}{dx} = u\dfrac{du}{dy}$. The differential equation (DE) becomes a first-order separable equation:

$$y \cdot u\frac{du}{dy} + u^2 - u = 0 \;\;\Rightarrow\;\; \frac{du}{dy} = \frac{u - u^2}{yu} \;\;\Rightarrow\;\; \int \frac{1}{1-u}\,du = \int \frac{1}{y}\,dy,$$

and so for any $c_0 \in \mathbb{R}$ we have

$$\ln|1-u| = -\ln|y| + c_0 \;\;\Rightarrow\;\; |1-u| = e^{c_0}e^{-\ln|y|} = \frac{e^{c_0}}{|y|} \;\;\Rightarrow\;\; |y(1-u)| = e^{c_0},$$

and so $y(1-u) = \pm e^{c_0}$. Letting $c_1 = \pm e^{c_0}$, so that $c_1 \neq 0$, we get

$$y(1-u) = c_1 \;\;\Rightarrow\;\; \frac{dy}{dx} = u = 1 - \frac{c_1}{y}.$$

This again is separable, yielding

$$\int \frac{y}{y-c_1}\,dy = \int dx \;\;\Rightarrow\;\; \int \left(1 + \frac{c_1}{y-c_1}\right) dy = \int dx,$$

and hence

$$y + c_1 \ln|y - c_1| = x + c_2$$

for $c_1 \neq 0$ and $c_2 \in \mathbb{R}$. However, if we set $c_1 = 0$ we find from this that $y = x + c_2$, which also satisfies the DE. Moreover we find that $y = c$, c any constant, also satisfies the DE. Thus there is a two-parameter family of solutions and a one-parameter family:

$$y + c_1 \ln|y - c_1| = x + c_2, \quad c_1, c_2 \in \mathbb{R}; \qquad y = c, \quad c \in \mathbb{R}.$$

With some algebra we can eliminate the absolute value in the two-parameter family and recast it differently as

$$y = C_1 + C_2 e^{(x-y)/C_1}, \quad C_1, C_2 \neq 0.$$

2 In general we have

$$y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y(4)(0)}{4!}x^4 + \cdots.$$

Now,

$$\begin{aligned} y'' = 4x + 2y^2 \;\;&\Rightarrow\;\; y''(0) = 2y^2(0) = 2(-1)^2 = 2, \\ y''' = 4 + 4yy' \;\;&\Rightarrow\;\; y'''(0) = 4 + 4y(0)y'(0) = 4 + 4(-1)(2) = -4, \\ y^{(4)} = 4(y')^2 + 4yy'' \;\;&\Rightarrow\;\; y^{(4)}(0) = 4(2)^2 + 4(-1)(2) = 8. \end{aligned}$$

Therefore

$$y \approx -1 + 2x + x^2 - \frac{2}{3}x^3 + \frac{1}{3}x^4.$$

3a Auxiliary equation is $r^2 + 4 = 0$, so that $r = \pm 2i$ and hence the general solution to the corresponding homogeneous DE is

$$y_h = c_1 \cos 2x + c_2 \sin 2x.$$

A particular solution to the DE $y'' + 4y = 4\cos x + 3\sin x$ will have the form

$$y_{p_1} = A\cos x + B\sin x.$$

Putting this into the DE we find we must have $A = \frac{4}{3}$ and $B = 1$. A particular solution to the DE $y'' + 4y = -8$ will have the form $y_{p_2} = C$, and just by inspection we can see that $y_{p_2} = -2$ works. By the Superposition Principle, therefore,

$$y_p = y_{p_1} + y_{p_2} = \frac{4}{3}\cos x + \sin x - 2$$

is a particular solution to the original DE.

3b General solution is

$$y = \frac{4}{3}\cos x + \sin x - 2 + c_1\cos 2x + c_2\sin 2x.$$

3c With $y(0) = 0$ we get $c_1 = \frac{2}{3}$, and with $y'(0) = -2$ we get $c_2 = -\frac{3}{2}$. The solution to the IVP is therefore

$$y = \frac{4}{3}\cos x + \sin x - 2 + \frac{2}{3}\cos 2x - \frac{3}{2}\sin 2x.$$

4 Auxiliary equation $r^2+2r+1 = 0$ has double root $r = -1$, so $y_1 = e^{-x}$ and $y_2 = xe^{-x}$ form a fundamental set for the DE, and $y_h = c_1e^{-x}+c_2xe^{-x}$ is the general solution to the corresponding homogeneous DE. A particular solution to the original DE has form $y_p = u_1(x)e^{-x}+u_2(x)xe^{-x}$, and with $q(x) = e^{-x}/x$ we have

$$u_1(x) = -\int \frac{y_2(x)q(x)}{\mathcal{W}[y_1, y_2]}\,dx = -\int \frac{xe^{-x}\cdot e^{-x}/x}{e^{-2x}}\,dx = -\int dx = -x$$

and

$$u_2(x) = \int \frac{y_1(x)q(x)}{\mathcal{W}[y_1, y_2]}\,dx = \int \frac{1}{x}\,dx = \ln|x|.$$

So $y_p = (\ln|x| - 1)xe^{-x}$, and the general solution is

$$y = \frac{(\ln|x| - 1)x + c_1 + c_2x}{e^x}.$$