Math 250 Exam #2 Key (Fall 2020)

1 Start with $\dfrac{dT}{dt} = k(T - M)$ with $T(0) = 200$ and $T(1) = 190$. Also $M = 72$. This gives an equation that is separable:

$$\int \frac{1}{T-72}\,dT = \int k\,dt \quad\Rightarrow\quad \ln|T-72| = kt + C \quad\Rightarrow\quad T(t) = 72 + Ce^{kt}.$$

With the initial conditions we find $C = 128$ and $k = \ln(59/64) \approx -0.0813$, and so

$$T(t) = 72 + 128e^{-0.0813t}.$$

Finally we find t such that $T(t) = 120$, a precalculus problem which solves to give $t \approx 12.1$ minutes.

2 Let $x(t)$ be the amount of dye (in grams) in the tank at time t. Then $x(0) = 200$ g. Now,

$$\frac{dx}{dt} = -\frac{x}{100} \quad\Rightarrow\quad \ln x = -\frac{t}{100} + c \quad\Rightarrow\quad x(t) = Ce^{-t/100},$$

with $x(0) = 200$ implying that $C = 200$, and so $x(t) = 200e^{-t/100}$. Now we find t such that $x(t) = 0.01x(0) = 2$ g, which solves to give $t \approx 460.5$ minutes.

3 It's a little faster using the definition instead of the Wronskian: Suppose

$$a_1x + a_2x^{-2} + a_3x^{-2}\ln x = 0$$

for all $x \in (0,\infty)$. Letting x be 1, e, and $1/e$, say, gives the system of equations

$$\begin{cases} a_1 + a_2 = 0 \\ ea_1 + e^{-2}a_2 + e^{-2}a_3 = 0 \\ e^{-1}a_1 + e^2a_2 - e^2a_3 = 0 \end{cases} \quad \text{or} \quad \begin{cases} a_1 + a_2 = 0 \\ e^3a_1 + a_2 + a_3 = 0 \\ a_1 + e^3a_2 - e^3a_3 = 0 \end{cases}$$

The first equation gives $a_2 = -a_1$. Putting this into the other two equations in the system at right, and then adding those equations, yields $(1 - e^3)a_3 = 0$ and hence $a_3 = 0$. From this we can quickly find that $a_1 = 0$ and $a_2 = 0$ as well, and therefore $\{x, x^{-2}, x^{-2}\ln x\}$ is a linearly independent set of functions on $(0,\infty)$. The general solution to the differential equation is

$$y = c_1x + c_2x^{-2} + c_3x^{-2}\ln x$$

for arbitrary parameters c_1, c_2, c_3.

The Wronskian, if computed, works out to $9x^{-6}$, which is clearly nonzero for any $x \in (0,\infty)$, again proving the linear independence of the functions according to a theorem we've seen.

4 Get the standard form first:

$$y'' + \frac{2x+2}{1-2x-x^2}y' - \frac{2}{1-2x-x^2}y = 0.$$

Now, with the supplied formula,

$$y_2(x) = (x+1)\int \frac{\exp\left(-\int \frac{2x+2}{1-2x-x^2}\,dx\right)}{(x+1)^2}\,dx$$

Letting $u = 1 - 2x - x^2$, so that $-du = (2x+2)\,dx$, we get

$$\int \frac{2x+2}{1-2x-x^2}\,dx = -\int \frac{1}{u}\,du = -\ln|u| = -\ln|1-2x-x^2|.$$

Thus, at least for $x < -1-\sqrt{2}$ or $x > -1+\sqrt{2}$ (where $1-2x-x^2 < 0$ is assured) we have

$$\exp\left(-\int \frac{2x+2}{1-2x-x^2}\,dx\right) = e^{\ln|1-2x-x^2|} = |1-2x-x^2| = x^2+2x-1,$$

and therefore

$$y_2(x) = (x+1)\int \frac{x^2+2x-1}{(x+1)^2}\,dx = (x+1)\int \left(1 - \frac{2}{(x+1)^2}\right)dx = x^2+x+2.$$

5a Auxiliary equation is $2r^2 - 7r + 3 = 0$, which has distinct real roots $r = \frac{1}{2}, 3$. The general solution is therefore $y = c_1 e^{x/2} + c_2 e^{3x}$.

5b Auxiliary equation is $2r^3 - 5r^2 + 8r - 20 = 0$, which has roots $r = \frac{5}{2}$ and $r = \pm 2i$ since

$$2r^3 - 5r^2 + 8r - 20 = r^2(2r-5) + 4(2r-5) = (2r-5)(r^2+4).$$

The general solution is therefore

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 e^{5x/2}.$$

6 Auxiliary equation is $r^2 - 2r + 1 = 0$, so $r = 1$ is a double root, and the general solution must be $y = c_1 e^x + c_2 x e^x$. With $y(0) = 5$ we immediately get $c_1 = 5$, then with $y'(0) = 10$ and $y' = (5 + c_2)e^x + c_2 x e^x$ we get $c_2 = 5$. Solution to the IVP is therefore

$$y = 5e^x + 5xe^x.$$