Math 250 Exam #1 Key (Fall 2020)

1 With y as given and $y(0) = 1$ we obtain $c_1 + c_2 = 1$. With

$$y' = 3c_1e^{3x} - c_2e^{-x} - 2$$

and $y'(0) = -3$ we obtain $3c_1 - c_2 = -1$. Solving the system of equations yields $c_1 = 0$ and $c_2 = 1$, so that $y = e^{-x} - 2x$ is a solution to the IVP.

2 From

$$\int y\,dy = \int \frac{x^2}{1+x^3}\,dx$$

we obtain $\frac{1}{2}y^2 = \frac{1}{3}\ln|1+x^3| + C$ for $-\infty < C < \infty$, so that

$$|1+x^3| = Ce^{3y^2/2}$$

for $C > 0$, and therefore

$$1 + x^3 = Ce^{3y^2/2}$$

for $C \neq 0$. (In this case y cannot be isolated without reintroducing logarithms and absolute values.)

3 The equation is separable, giving

$$\int (2y - \sin y)\,dy = \int (\sin x - x)\,dx \quad\Rightarrow\quad y^2 + \cos y = -\cos x - \frac{1}{2}x^2 + C.$$

With the initial condition $y(0) = 0$ we get $C = 2$, and therefore

$$y^2 + \cos y + \cos x + \frac{x^2}{2} = 2.$$

Isolating y would reintroduce an absolute value that cannot be resolved even with the given initial condition given, so we stop here. Remember: $\sqrt{y^2} = |y|$.

4 Standard form is

$$y' - \frac{3}{2x}y = \frac{9}{2}x^2,$$

and so an integrating factor is

$$\mu(x) = \exp\left(-\frac{3}{2}\int \frac{1}{x}\,dx\right) = \exp\left(-\frac{3}{2}\ln x\right) = x^{-3/2}.$$

Multiplying by $\mu(x)$ yields $x^{-3/2}y' - \frac{3}{2}x^{-5/2}y = \frac{9}{2}x^{1/2}$, or $(x^{-3/2}y)' = \frac{9}{2}x^{1/2}$. Integrating:

$$x^{-3/2}y = 3x^{3/2} + C \quad\Rightarrow\quad y = 3x^3 + Cx^{3/2}.$$

5 There is a function $F(x,y)$ such that $F_x = x + \tan^{-1} y$ and $F_y = \dfrac{x+y}{1+y^2}$. Now,

$$F(x,y) = \int (x + \tan^{-1} y)\,dx = \frac{1}{2}x^2 + x\tan^{-1} y + g(y)$$

for arbitrary function $g(y)$. However,

$$F_y = \frac{x+y}{1+y^2} \;\Rightarrow\; \frac{x}{1+y^2} + g'(y) = \frac{x+y}{1+y^2} \;\Rightarrow\; g(y) = \int \frac{y}{1+y^2}\,dy = \frac{1}{2}\ln|1+2y|,$$

and so the one-parameter family of solutions $F(x,y) = C$ may be expressed as

$$\frac{1}{2}x^2 + x\tan^{-1} y + \frac{1}{2}\ln|1+2y| = C.$$

While it is possible to eliminate the absolute value, in this case the resultant expression would be too unwieldy. (Yes, this is a subjective gray area.)

6 We have

$$\frac{dy}{dx} = -\frac{4+3(y/x)}{2+(y/x)},$$

so let $v = y/x$, implying $y' = v + xv'$ and hence

$$v + xv' = -\frac{4+3v}{2+v}.$$

The equation is separable: applying partial fraction decomposition we find that

$$-\int \frac{v+2}{v^2+5v+4}\,dv = \int \frac{1}{x}\,dx \;\Rightarrow\; -\int\left(\frac{2/3}{v+4} + \frac{1/3}{v+1}\right)dv = \ln|x| + C.$$

Thus

$$-\frac{2}{3}\ln\left|\frac{y}{x}+4\right| - \frac{1}{3}\ln\left|\frac{y}{x}+1\right| = -3\ln|x| + C.$$

Multiply by -3 and combine logarithms:

$$\ln\left[\left(\frac{y}{x}+4\right)^2\left|\frac{y}{x}+1\right|\right] = \ln|x|^{-3} + C.$$

Exponentiate:

$$\left(\frac{y}{x}+4\right)^2\left|\frac{y}{x}+1\right| = C|x|^{-3}, \quad C > 0.$$

Multiply by $|x|^3$:

$$\left(\frac{y}{x}+4\right)^2 |x^2y + x^3| = C, \quad C > 0.$$

Remove absolute values and expand the domain for C accordingly:

$$\left(\frac{y}{x}+4\right)^2 (x^2y + x^3) = C, \quad C \neq 0.$$

Alternative form:

$$(y+4x)^2(y+x) = C, \quad C \neq 0.$$

If $C = 0$ the last form implies either $y = -x$ or $y = -4x$. Both can be seen to satisfy the original differential equation, and so $C = 0$ can be included to obtain

$$(y+4x)^2(y+x) = C, \quad C \in (-\infty, \infty).$$