MATH 250 EXAM #5 KEY (FALL 2008)

1a $\mathcal{L}\{9-6e^{-2t}+e^{-4t}\} = \frac{9}{s} - 6\cdot\frac{1}{s+2} + \frac{1}{s+4} = \frac{9}{s} - \frac{6}{s+2} + \frac{1}{s+4}$

1b $\mathcal{L}\{\frac{1}{2}\sin 0 + \frac{1}{2}\sin 6t\} = \frac{1}{2}\mathcal{L}\{\sin 6t\} = \frac{1}{2}\cdot\frac{6}{s^2+6^2} = \frac{3}{s^2+36}$

2a $2s^2+s+6 = 2(s^2+\frac{1}{2}s+3) = 2\left[(s^2+\frac{1}{2}s+\frac{1}{16})+\frac{47}{16}\right] = 2\left[(s+\frac{1}{4})^2+\frac{47}{16}\right]$, so: $\mathcal{L}^{-1}\left\{\frac{s-1}{2[(s+1/4)^2+47/16]}\right\}$

$= \frac{1}{2}\left[\mathcal{L}^{-1}\left\{\frac{s+1/4}{(s+1/4)^2+47/16}\right\} - \frac{5}{4}\cdot\frac{4}{\sqrt{47}}\mathcal{L}^{-1}\left\{\frac{\sqrt{47}/4}{(s+1/4)^2+47/16}\right\}\right] = \frac{1}{2}e^{-t/4}\cos\frac{\sqrt{47}}{4}t - \frac{5}{2\sqrt{47}}e^{-t/4}\sin\frac{\sqrt{47}}{4}t$

2b $\frac{s-10}{(s+1)(s-6)} = \frac{A}{s+1}+\frac{B}{s-6} \Rightarrow s-10 = A(s-6)+B(s+1) = (A+B)s+(-6A+B) \Rightarrow A+B=1 \;\&\; -6A+B=-10 \Rightarrow$

$A = 11/7 \;\&\; B=-4/7$, so $\mathcal{L}^{-1}\left\{\frac{s-10}{(s+1)(s-6)}\right\} = \mathcal{L}^{-1}\left\{\frac{11/7}{s+1}+\frac{-4/7}{s-6}\right\} = \frac{11}{7}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{4}{7}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}$

$= \frac{11}{7}e^{-t} - \frac{4}{7}e^{6t}$

3 $\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 4\cdot\frac{1}{s-3} \Rightarrow (s^2Y - sy(0) - y'(0)) - 4(sY-y(0)) + 5Y = \frac{4}{s-3} \Rightarrow$

$(s^2Y - 2s - 7) - 4sY + 8 + 5Y = \frac{4}{s-3} \Rightarrow (s^2-4s+5)Y = (2s-1) + \frac{4}{s-3} \Rightarrow Y(s) = \frac{2s-1}{s^2-4s+5} + \frac{4}{(s-3)(s^2-4s+5)}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{2s-1}{(s-2)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s-3}+\frac{-2s+2}{s^2-4s+5}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s-1/2}{(s-2)^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$

$-2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-2)^2+1}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1^2}+\frac{3/2}{(s-2)^2+1^2}\right\} + 2e^{3t} - 2\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1^2}+\frac{1}{(s-2)^2+1^2}\right\}$

$= \cancel{2e^{2t}\cos t} + 3e^{2t}\sin t + 2e^{3t} - \cancel{2e^{2t}\cos t} - 2e^{2t}\sin t$

$\therefore\; y(t) = e^{2t}\sin t + 2e^{3t}$ ✓

4 $(s^2Y - sy(0) - y'(0)) + 5(sY - y(0)) + 6Y = \mathcal{L}\{t\,u(t-2)\} = e^{-2s}\mathcal{L}\{t+2\} = e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right) \Rightarrow$

$s^2Y - 1 + 5sY + 6Y = e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right) \Rightarrow (s^2+5s+6)Y = e^{-2s}(1/s^2 + 2/s) + 1 \Rightarrow$

$Y(s) = \frac{e^{-2s}(1/s^2+2/s)+1}{s^2+5s+6} = \frac{e^{-2s}}{s^2(s+2)(s+3)} + \frac{2e^{-2s}}{s(s+2)(s+3)} + \frac{1}{(s+2)(s+3)} \Rightarrow$

$y(t) = \mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{-5/36}{s}+\frac{1/6}{s^2}+\frac{1/4}{s+2}+\frac{-1/9}{s+3}\right) + e^{-2s}\left(\frac{1/3}{s}+\frac{-1}{s+2}+\frac{2/3}{s+3}\right) + \frac{1}{s+2}+\frac{-1}{s+3}\right\}$

$= \mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{7/36}{s}+\frac{1/6}{s^2}+\frac{-3/4}{s+2}+\frac{5/9}{s+3}\right) + \frac{1}{s+2} - \frac{1}{s+3}\right\}$

$= \frac{7}{36}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} - \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+2}\right\} + \frac{5}{9}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+3}\right\} + e^{-2t} - e^{-3t}$

$= \frac{7}{36}u(t-2) + \frac{1}{6}(t-2)u(t-2) - \frac{3}{4}e^{-2(t-2)}u(t-2) + \frac{5}{9}e^{-3(t-2)}u(t-2) + e^{-2t} - e^{-3t}$

$= \left[\frac{7}{36} + \frac{1}{6}(t-2) - \frac{3}{4}e^{-2t+4} + \frac{5}{9}e^{-3t+6}\right]u(t-2) + e^{-2t} - e^{-3t}$

$= \left(\frac{1}{6}t - \frac{5}{36} - \frac{3}{4}e^{-2t+4} + \frac{5}{9}e^{-3t+6}\right)u(t-2) + e^{-2t} - e^{-3t}$ ✓

$\boxed{5}$ $(s^2Y - sy(0) - y'(0)) + 2(sY - y(0)) + 2Y = \mathcal{L}\{\delta(t-\pi)\} \Rightarrow$

$s^2Y - s - 1 + 2sY - 2 + 2Y = e^{-\pi s} \Rightarrow$

$(s^2+2s+2)Y = s + 3 + e^{-\pi s} \Rightarrow Y(s) = \dfrac{s+3}{s^2+2s+2} + \dfrac{1}{s^2+2s+2}e^{-\pi s} \Rightarrow$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+3}{(s+1)^2+1^2} + \frac{1}{(s+1)^2+1^2}e^{-\pi s}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1^2} + 2\cdot\frac{1}{(s+1)^2+1^2}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+1^2}\right\}$$

$$y(t) = e^{-t}\cos t + 2e^{-t}\sin t + e^{-(t-\pi)}\sin(t-\pi)\,u(t-\pi)$$

$\boxed{6}$ Let $y(t)$ = Amt. of salt in tank at time t, so $y(0) = 240$ kg

$$\frac{dy}{dt} = g(t) - \left(\frac{16\text{ L}}{\text{min}}\right)\left(\frac{y(t)\text{ kg}}{800\text{ L}}\right), \text{ where } g(t) = \begin{cases}\left(\frac{16\text{ L}}{\text{min}}\right)\left(\frac{0.6\text{ kg}}{\text{L}}\right), & t \le 20\\ \left(\frac{16\text{ L}}{\text{min}}\right)\left(\frac{0.2\text{ kg}}{\text{L}}\right), & t > 20\end{cases} \Rightarrow g(t) = \begin{cases}9.6, & t \le 20\\ 3.2, & t > 20\end{cases}$$

$$\Rightarrow \frac{dy}{dt} = 9.6 - 6.4u(t-20) - 0.02y \Rightarrow y' + 0.02y = 9.6 - 6.4u(t-20) \Rightarrow$$

$$sY - y(0) + 0.02Y = \mathcal{L}\{9.6 - 6.4u(t-20)\} \Rightarrow sY - 240 + 0.02Y = \frac{9.6}{s} - \frac{6.4e^{-20s}}{s} \Rightarrow$$

$$Y(s) = \frac{9.6}{s(s+0.02)} - \frac{6.4e^{-20s}}{s(s+0.02)} + \frac{240}{s+0.02} \Rightarrow$$

$$y(t) = 9.6\mathcal{L}^{-1}\left\{\frac{50}{s} + \frac{-50}{s+0.02}\right\} - 6.4\mathcal{L}^{-1}\left\{\frac{1}{s(s+0.02)}e^{-20s}\right\} + 240e^{-0.02t}$$

$$= 9.6\left[50 - 50e^{-0.02t}\right] - 6.4\left[50 - 50e^{-0.02(t-20)}\right]u(t-20) + 240e^{-0.02t}$$

$$= 480(1 - e^{-0.02t}) - 320(1 - e^{-0.02(t-20)})u(t-20) + 240e^{-0.02t}$$

$$= 480 - 240e^{-0.02t} - 320(1 - e^{-0.02(t-20)})u(t-20)$$

Concentration $c(t) = \dfrac{y(t)}{800}$