MATH 250 EXAM #4 KEY (FALL 2008)

1 $y_p(\theta) = A\theta^s e^{-\theta}\cos\theta + B\theta^s e^{-\theta}\sin\theta$, so $\alpha + i\beta = -1+i$. Aux. eq. is $x^2+2x+2=0 \Rightarrow x = \frac{-2\pm\sqrt{2^2-4(2)}}{2}$ $= \frac{-2\pm 2i}{2} = -1\pm i$, so $s=1$. Now $y_p = \theta e^{-\theta}(A\cos\theta + B\sin\theta)$

$$y_p' = (e^{-\theta} - \theta e^{-\theta})(A\cos\theta + B\sin\theta) + \theta e^{-\theta}(-A\sin\theta + B\cos\theta)$$
$$= e^{-\theta}\left[(A - A\theta + B\theta)\cos\theta + (B - B\theta - A\theta)\sin\theta\right]$$
$$y_p'' = e^{-\theta}\left[(-A+B)\cos\theta - (A - A\theta + B\theta)\sin\theta + (-B-A)\sin\theta + (B - B\theta - A\theta)\cos\theta\right]$$
$$- e^{-\theta}\left[(A - A\theta + B\theta)\cos\theta + (B - B\theta - A\theta)\sin\theta\right]$$

So $y_p'' + 2y' + 2y = e^{-\theta}\cos\theta \Rightarrow B\cos\theta - A\sin\theta - A\sin\theta + B\cos\theta - B\theta\cos\theta - B\theta\cos\theta + 2B\theta\cos\theta = \cos\theta \Rightarrow$
$2B\cos\theta - 2A\sin\theta = \cos\theta \Rightarrow B = 1/2$ & $A = 0$. Thus $y_p(\theta) = \frac{\theta}{2}e^{-\theta}\sin\theta$

General Solution: $y(\theta) = y_p(\theta) + y_h(\theta) = \frac{\theta}{2}e^{-\theta}\sin\theta + C_1 e^{-\theta}\cos\theta + C_2 e^{-\theta}\sin\theta$

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2 $y'' + y' - 12y = e^t$: $y_{p_1} = Ae^t$, so $Ae^t + Ae^t - 12Ae^t = e^t \Rightarrow -10Ae^t = e^t \Rightarrow A = -\frac{1}{10} \Rightarrow y_{p_1} = -\frac{1}{10}e^t$

$y'' + y' - 12y = e^{2t}$: $y_{p_2} = Ae^{2t}$, so $4Ae^{2t} + 2Ae^{2t} - 12Ae^{2t} = e^{2t} \Rightarrow -6Ae^{2t} = e^{2t} \Rightarrow A = -\frac{1}{6} \Rightarrow$
$y_{p_2} = -\frac{1}{6}e^{2t}$

$y'' + y' - 12y = -1$: $y_{p_3} = A$, so $0 + 0 - 12A = -1 \Rightarrow A = \frac{1}{12} \Rightarrow y_{p_3} = \frac{1}{12}$

$y'' + y' - 12y = 0$: $y_h(t) = C_1 e^{-4t} + C_2 e^{3t}$

So $y(t) = -\frac{1}{10}e^t - \frac{1}{6}e^{2t} + \frac{1}{12} + C_1 e^{-4t} + C_2 e^{3t}$ & $y'(t) = -\frac{1}{10}e^t - \frac{1}{3}e^{2t} - 4C_1 e^{-4t} + 3C_2 e^{3t}$

$$\left.\begin{array}{l} \text{Now, } y(0)=1 \text{ gives: } 1 = -\frac{1}{10} - \frac{1}{6} + \frac{1}{12} + C_1 + C_2 \Rightarrow C_1 + C_2 = \frac{71}{60} \\ \quad y'(0) = 3 \text{ gives: } 3 = -\frac{1}{10} - \frac{1}{3} - 4C_1 + 3C_2 \Rightarrow -4C_1 + 3C_2 = \frac{103}{30} \end{array}\right\} \text{So } C_1 = 1/60 \text{ \& } C_2 = 7/6$$

$\therefore y(t) = -\frac{1}{10}e^t - \frac{1}{6}e^{2t} + \frac{1}{12} + \frac{1}{60}e^{-4t} + \frac{7}{6}e^{3t}$ ✓

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3 $y'' + 3y' + 6.25y = 0$, $y(0) = 0.25$, $y'(0) = -2$

Aux. eq. $r^2 + 3r + 6.25 = 0 \to r = -\frac{3}{2} \pm 2i$, so $y(t) = e^{-3/2 t}(C_1\cos 2t + C_2\sin 2t)$

Then $y'(t) = -\frac{3}{2}e^{-3/2 t}(C_1\cos 2t + C_2\sin 2t) + e^{-3/2 t}(-2C_1\sin 2t + 2C_2\cos 2t)$

$y(0) = 0.25$: $C_1\cdot 1 + C_2\cdot 0 = 0.25 \Rightarrow C_1 = 0.25$

$y'(0) = -2$: $-\frac{3}{2}(C_1 + 0) + (0 + 2C_2) = -2 \Rightarrow -\frac{3}{2}C_1 + 2C_2 = -2 \Rightarrow C_2 = -\frac{13}{16}$

So $y(t) = e^{-\frac{3}{2}t}\left(\frac{1}{4}\cos 2t - \frac{13}{16}\sin 2t\right)$ ✓

Find t when $y(t) = 0$: $\frac{1}{4}\cos 2t - \frac{13}{16}\sin 2t = 0 \Rightarrow \tan 2t = \frac{4}{13} \Rightarrow 2t = \arctan\left(\frac{4}{13}\right)$
$\Rightarrow t = \frac{1}{2}\arctan\left(\frac{4}{13}\right) = 0.149$ sec. ✓

$\boxed{4}$ From #3 we have $y(t) = e^{-\frac{3}{2}t}\left(\frac{1}{4}\cos 2t - \frac{13}{16}\sin 2t\right)$, so find Δt such that $2(t+\Delta t) = 2t + 2\pi$:
$2t + 2(\Delta t) = 2t + 2\pi \Rightarrow \Delta t = \pi$, so Quasiperiod $= \pi$ sec. & Quasifrequency $= \frac{1}{\pi}$ cycle/sec.

$\boxed{5}$ $0.5y'' + 0.4y' + 8y = 0$, $y(0) = -1$, $y'(0) = 0$

Aux. eq. $0.5r^2 + 0.4r + 8 = 0 \Rightarrow r = \dfrac{-0.4 \pm \sqrt{0.4^2 - 4(0.5)(8)}}{2(0.5)} = -0.4 \pm \sqrt{-15.84} = -0.4 \pm 3.98$

So $y(t) = e^{-0.4t}(C_1 \cos 3.98t + C_2 \sin 3.98t)$

From $y(0) = -1$ we get $C_1 \cdot 1 + C_2 \cdot 0 = -1 \Rightarrow C_1 = -1$

Also $y'(t) = -0.4e^{-0.4t}(C_1 \cos 3.98t + C_2 \sin 3.98t) + e^{-0.4t}(-3.98 C_1 \sin 3.98t + 3.98 C_2 \cos 3.98t)$

From $y'(0) = 0$ we get $-0.4(C_1 + 0) + 1 \cdot (0 + 3.98 C_2 \cdot 1) = 0 \Rightarrow 0.4 + 3.98C_2 = 0 \Rightarrow C_2 = -0.101$

$\therefore y(t) = e^{-0.4t}(-\cos 3.98t - 0.101 \sin 3.98t)$ ✓

Find t for which $y'(t) = 0$: $t = 0.769$ sec. So max. displacement to right is $y(0.769) = 0.732$ m.

$\boxed{6}$ $8y'' + 3y' + 40y = 2\sin 2t + \overset{78.4}{mg}$, $y(0) = 1.96$ m, $y'(0) = 0$ m/s

Aux. Eq.: $8x^2 + 3x + 40 = 0 \Rightarrow x = \dfrac{-3 \pm \sqrt{9 - 4(8)(40)}}{16} = -\frac{3}{16} \pm 2.23i \approx -0.1875 \pm 2.228i$

For $8y'' + 3y' + 40y = 2\sin 2t$, $y_p(t) = A\cos 2t + B\sin 2t$. Then $y_p'(t) = -2A\sin 2t + 2B\cos 2t$
& $y_p''(t) = -4A\cos 2t - 4B\sin 2t$. Now...

$8y_p'' + 3y_p' + 40y_p = 2\sin 2t \Rightarrow -32A\cos 2t - 32B\sin 2t - 6A\sin 2t + 6B\cos 2t + 40A\cos 2t + 40B\sin 2t = 2\sin 2t$

$\Rightarrow (-32A + 6B + 40A)\cos 2t + (-32B - 6A + 40B)\sin 2t = 2\sin 2t$

So $8A + 6B = 0$ & $8B - 6A = 2$. Hence $A = -3/25$ & $B = 4/25$

$y_p(t) = -\frac{3}{25}\cos 2t + \frac{4}{25}\sin 2t$

For $8y'' + 3y' + 40y = 78.4$, $\bar{y}_p(t) = \dfrac{78.4}{40} \approx 1.96$

General Solution: $y(t) = 1.96 - \frac{3}{25}\cos 2t + \frac{4}{25}\sin 2t + e^{-3/16 t}(C_1 \cos 2.228t + C_2 \sin 2.228t)$

Steady-state solution: $y_s(t) = \lim_{t\to\infty} y(t) \Rightarrow 1.96 - \frac{3}{25}\cos 2t + \frac{4}{25}\sin 2t$

Freq. $= \frac{1}{\pi}$ cycle/sec (see #4), Amplitude $= 0.4$ m ✓

$\boxed{7}$ $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty t e^{3t} e^{-st}\, dt = \int_0^\infty t e^{(3-s)t}\, dt$

Let $u = t$, $dv = e^{(3-s)t}\, dt$; so $du = dt$, $v = \int e^{(3-s)t}\, dt = \frac{1}{3-s} e^{(3-s)t}$ ...

$$F(s) = \frac{t}{3-s} e^{(3-s)t} \Big|_0^\infty - \int_0^\infty \frac{1}{3-s} e^{(3-s)t}\, dt = \lim_{N\to\infty} \left[\frac{N}{3-s} e^{(3-s)N} - 0\right] - \lim_{N\to\infty} \frac{1}{3-s} \int_0^N e^{(3-s)t}\, dt$$

$$= 0 - \frac{1}{(3-s)^2} \lim_{N\to\infty} \left[e^{(3-s)t}\right]_0^N = -\frac{1}{(3-s)^2} \lim_{N\to\infty} \left(e^{(3-s)N} - 1\right) = \frac{1}{(3-s)^2}, \quad s > 3.$$

$\boxed{8}$ $\mathcal{L}\{f(t)\} = \int_0^1 (1-t) e^{-st}\, dt + \lim_{N\to\infty} \int_1^N 0 \cdot e^{-st}\, dt = \int_0^1 e^{-st}\, dt - \int_0^1 t e^{-st}\, dt$

$$= -\frac{1}{s}\left[e^{-st}\right]_0^1 - \left\{\left[-\frac{t}{s} e^{-st}\right]_0^1 + \int_0^1 \frac{1}{s} e^{-st}\, dt\right\}$$

$$= -\frac{1}{s}\left(e^{-s} - 1\right) - \left\{-\frac{1}{s} e^{-s} + \frac{1}{s}\left[-\frac{1}{s} e^{-st}\right]_0^1\right\}$$

$$= \frac{1}{s} + \frac{1}{s^2}\left[e^{-st}\right]_0^1 = \frac{1}{s} + \frac{1}{s^2}\left(e^{-s} - 1\right) = \frac{1}{s} + \frac{e^{-s} - 1}{s^2}$$

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