MATH 250 EXAM #3 KEY (FALL 2008)

1 $\frac{dT}{dt} = K[M(t) - T(t)] + H(t) + U(t) = \frac{1}{4}[39 - T(t)] + 0 + 0 \Rightarrow \frac{dT}{dt} = \frac{1}{4}(39 - T) \Rightarrow$

$\int \frac{1}{39-T}\,dT = \int \frac{1}{4}\,dt \; = -\ln(39-T) = \frac{t}{4} + C \Rightarrow \; 39 - T = e^{-t/4 + C} \Rightarrow T(t) = 39 - Ce^{-t/4}$

Let $t=0$ at noon. Then $T(0) = 24$, so we get $24 = 39 - C \Rightarrow C = 15$, and thus $T(t) = 39 - 15e^{-t/4}$.

- At 1:30 pm temp. is $T(1.5) = 39 - 15e^{-1.5/4} = 28.7\ °C$
- Next, $36 = 39 - 15e^{-t/4} \Rightarrow e^{-t/4} = 0.2 \Rightarrow -t/4 = \ln 0.2 \Rightarrow t = 6.44$ hr., so temp. will reach 36°C at 6:26 pm ✓

2 Let $M(t) = 64.5 - 17.5\cos\left(\frac{\pi}{12}t\right)$, so $M(0) = 47$, $M(12) = 82$, etc. — where we let $t=0$ at 2:00 am on Day 1, $t=12$ at 2:00 pm on Day 1, $t=24$ at 2:00 am on Day 2, and so on. Now: $\frac{dT}{dt} = K[M(t) - T(t)] = \frac{1}{3}\left[64.5 - 17.5\cos\left(\frac{\pi}{12}t\right) - T(t)\right]$, or from Eq. (4) in notes...

$$T(t) = e^{-t/3}\int e^{t/3}\cdot\frac{1}{3}M(t)\,dt + Ce^{-t/3} = e^{-t/3}\cdot\frac{1}{3}\int\left(64.5e^{t/3} - 17.5e^{t/3}\cos\left(\tfrac{\pi}{12}t\right)\right)dt + \underbrace{Ce^{-t/3}}_{\text{dies off...}}$$

$$\approx \frac{e^{-t/3}}{3}\left[64.5\int e^{t/3}dt - 17.5\int e^{t/3}\cos\left(\tfrac{\pi}{12}t\right)dt\right] = \frac{e^{-t/3}}{3}\left\{64.5(3e^{t/3}) - 17.5\left[\frac{e^{t/3}}{(1/3)^2 + (\pi/12)^2}\cdot\right.\right.$$

$$\left.\left.\left(\tfrac{1}{3}\cos\tfrac{\pi}{12}t + \tfrac{\pi}{12}\sin\tfrac{\pi}{12}t\right)\right]\right\} = 64.5 - \frac{35}{6}\left[\frac{144}{\pi^2+16}\left(\tfrac{1}{3}\cos\tfrac{\pi}{12}t + \tfrac{\pi}{12}\sin\tfrac{\pi}{12}t\right)\right]$$

$$= 64.5 - \frac{280}{\pi^2+16}\left(\cos\tfrac{\pi}{12}t + \tfrac{\pi}{4}\sin\tfrac{\pi}{12}t\right).$$

Set $T'(t) = 0$: $-\frac{280}{\pi^2+16}\left(-\frac{\pi}{12}\sin\frac{\pi}{12}t + \frac{\pi^2}{48}\cos\frac{\pi}{12}t\right) = 0 \Rightarrow \tan\frac{\pi}{12}t = \frac{\pi}{4} \Rightarrow$

$\frac{\pi}{12}t = \arctan\frac{\pi}{4} \Rightarrow t = \frac{12}{\pi}\arctan\frac{\pi}{4} \approx 2.54$ hr.

So temp. in garage reaches lowest & highest values at 4:32 am & 4:32 pm, resp.

3 $(-9A\cos 3t - 9B\sin 3t) + 2(-3A\sin 3t + 3B\cos 3t) + 4(A\cos 3t + B\sin 3t) = 5\sin 3t$

$(-5A\cos 3t + 6B\cos 3t) + (-6A\sin 3t - 5B\sin 3t) = 5\sin 3t$

$(-5A + 6B)\cos 3t + (-6A - 5B)\sin 3t = 5\sin 3t$

So $\begin{cases} -5A + 6B = 0 \\ -6A - 5B = 5 \end{cases} \Rightarrow B = -\frac{25}{61}$ & $A = -\frac{30}{61}$

$\boxed{4}$ $r^2 - 4r + 3 = 0 \Rightarrow (r-3)(r-1) = 0 \Rightarrow r = 1, 3,$ so $y(t) = c_1 e^{t} + c_2 e^{3t}$

$$\left.\begin{array}{l} y(0)=1: \quad c_1 + c_2 = 1 \\ y'(0)=\frac{1}{3}: \quad c_1 + 3c_2 = \frac{1}{3} \end{array}\right\} \text{So } c_1 = \frac{4}{3} \ \& \ c_2 = -\frac{1}{3}, \text{ so } y(t) = \frac{4}{3}e^{t} - \frac{1}{3}e^{3t}$$

$\boxed{5}$ $r^3 - 7r^2 + 7r + 15 = 0 \Rightarrow (r+1)(r^2 - 8r + 15) = 0 \Rightarrow (r+1)(r-5)(r-3) = 0 \Rightarrow r = -1, 3, 5$

$y(t) = c_1 e^{-t} + c_2 e^{3t} + c_3^{5t}$

$\boxed{6}$ $r^2 + 2r + 17 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4 - 4(17)}}{2} = \frac{-2 \pm \sqrt{-64}}{2} = \frac{-2 \pm 8i}{2} = -1 \pm 4i \Rightarrow$

$\alpha = -1$ & $\beta = 4 \Rightarrow y(t) = c_1 e^{-t}\cos 4t + c_2 e^{-t}\sin 4t$

$y(0) = 1: \quad c_1 = 1$

$y'(t) = e^{-t}(-4c_1 \sin 4t + 4c_2 \cos 4t) - e^{-t}(c_1 \cos 4t + c_2 \sin 4t)$

$\quad = (-4c_1 \sin 4t + 4c_2 \cos 4t - c_1 \cos 4t - c_2 \sin 4t)e^{-t}$

$\quad = [(4c_2 - c_1)\cos 4t + (-4c_1 - c_2)\sin 4t]e^{-t}$

$y'(0) = -1: \quad 4c_2 - c_1 = -1 \Rightarrow 4c_2 = c_1 - 1 \Rightarrow 4c_2 = 0 \Rightarrow c_2 = 0.$

$\therefore y(t) = e^{-t}\cos 4t$ ✓

$\boxed{7}$ $y_p(t) = t^1(A_1 t + A_0)\cos t + t^1(B_1 t + B_0)\sin t$ since $\alpha + i\beta = 0 + i1 = i$ is a root of $x^2 + 1 = 0$ (so $s = 1$).

Now, $y_p'' + y_p = 4t\cos t \Rightarrow (4B_1 t + 2A_1 + 2B_0)\cos t + (-4A_1 t - 2A_0 + 2B_1)\sin t = 4t\cos t \Rightarrow$

$\Rightarrow 4B_1 t\cos t + (2A_1 + 2B_0)\cos t - 4A_1 t\sin t + (-2A_0 + 2B_1)\sin t = 4t\cos t$

$\Rightarrow 4B_1 = 4, \quad 2A_1 + 2B_0 = 0, \quad -4A_1 = 0, \quad -2A_0 + 2B_1 = 0$

$\Rightarrow B_1 = 1, \quad B_0 = 0, \quad A_1 = 0, \quad A_0 = 1$

$\Rightarrow y_p(t) = t\cos t + t^2 \sin t$

$\boxed{8}$ $s = 0$ ✓ since $r = 1$ ✓ is not a root of $r^2 + 3r - 7 = 0$

$m = 4$ ✓

Get $y_p(t) = (A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0)e^{t}$ ✓