MATH 250 EXAM #2 KEY (FALL 2008)

1 $\frac{\partial M}{\partial y} = e^t + te^t = \frac{\partial N}{\partial t} \Rightarrow$ Exact! So there's some $F \ni \frac{\partial F}{\partial t} = M$ & $\frac{\partial F}{\partial y} = N$.

$\frac{\partial F}{\partial t} = M \Rightarrow F = \int (e^t y + te^t y)\,dt + g(y) = y(e^t + te^t - e^t) + g(y) = yte^t + g(y)$

$\frac{\partial F}{\partial y} = N \Rightarrow te^t + g'(y) = te^t + 2 \Rightarrow g'(y) = 2 \Rightarrow g(y) = 2y$

General solution is $F(t,y) = C$, or $yte^t + 2y = C$

$y(0) = -1$: $-1(0)e^0 + 2(-1) = C \Rightarrow C = -2$

$\therefore$ $\underline{\underline{yte^t + 2y = -2}}$ or $\underline{\underline{y = \frac{-2}{te^t + 2}}}$

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2 $\frac{\partial M}{\partial y} = 2x \neq -6x = \frac{\partial N}{\partial x} \Rightarrow$ Not exact, but $\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{-8x}{2xy} = -\frac{4}{y}$ implies $\mu(y) = e^{\int -4/y\,dy}$ $= e^{-4\ln y} = y^{-4}$ is an integrating factor. We get: $(2xy^{-3})\,dx + (y^{-2} - 3x^2y^{-4})\,dy = 0$, exact.

There's some $F \ni \frac{\partial F}{\partial x} = 2xy^{-3}$ & $\frac{\partial F}{\partial y} = y^{-2} - 3x^2y^{-4}$.

$\frac{\partial F}{\partial x} = 2xy^{-3} \Rightarrow F = \int 2xy^{-3}\,dx = x^2y^{-3} + g(y)$

$\frac{\partial F}{\partial y} = y^{-2} - 3x^2y^{-4} \Rightarrow -3x^2y^{-4} + g'(y) = y^{-2} - 3x^2y^{-4} \Rightarrow g'(y) = y^{-2} \Rightarrow g(y) = -y^{-1}$

General soln. is $F(x,y) = C$, or $x^2y^{-3} - y^{-1} = C$, or $\underline{\underline{x^2 - y^2 = Cy^3}}$

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3 $\underbrace{(2x^ny^{m+3} - 6x^{n+1}y^{m+1})}_{M}dx + \underbrace{(3x^{n+1}y^{m+2} - 4x^{n+2}y^m)}_{N}dy = 0$, need $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, or:

$2(m+3)x^ny^{m+2} - 6(m+1)x^{n+1}y^m = 3(n+1)x^ny^{m+2} - 4(n+2)x^{n+1}y^m$, which requires: $\begin{cases} 2(m+3) = 3(n+1) \\ -6(m+1) = -4(n+2) \end{cases}$

Thus $m = 9/5$ & $n = 11/5$, which gives us the exact eq:

$\underbrace{(2x^{11/5}y^{24/5} - 6x^{16/5}y^{14/5})}_{\hat{M}}dx + \underbrace{(3x^{16/5}y^{19/5} - 4x^{21/5}y^{9/5})}_{\hat{N}}dy = 0$. Find F so that $\frac{\partial F}{\partial x} = \hat{M}$, $\frac{\partial F}{\partial y} = \hat{N}$

$F = \int \hat{M}\,dx = 2\left(\frac{5}{16}\right)x^{16/5}y^{24/5} - 6\left(\frac{5}{21}\right)x^{21/5}y^{14/5} + g(y) = \frac{5}{8}x^{16/5}y^{24/5} - \frac{10}{7}x^{21/5}y^{14/5} + g(y)$

Now $\frac{\partial F}{\partial y} = \hat{N} \Rightarrow 3x^{16/5}y^{19/5} - 4x^{21/5}y^{9/5} + g'(y) = 3x^{16/5}y^{19/5} - 4x^{21/5}y^{9/5} \Rightarrow g'(y) = 0 \Rightarrow g(y) = A$

Solution: $F(x,y) = B$, or $\underline{\underline{\frac{5}{8}x^{16/5}y^{24/5} - \frac{10}{7}x^{21/5}y^{14/5} = C}}$ (here $C = B - A$).

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4 Let $v = y^{1-2} = y^{-1}$, so $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$. From eq. we have: $-y^{-2}\frac{dy}{dx} = -y^{-2}\cdot\left(\frac{2y}{x} - x^2y^2\right) \Rightarrow$

$\frac{dv}{dx} = -\frac{2y^{-1}}{x} + x^2 \Rightarrow \frac{dv}{dx} = -\frac{2}{x}v + x^2 \Rightarrow \frac{dv}{dx} + \frac{2}{x}v = x^2$, (1)

$\mu(x) = e^{\int 2/x\,dx} = e^{2\ln x} = x^2$, so (1) becomes: $x^2v' + 2xv = x^4 \Rightarrow (x^2v)' = x^4 \Rightarrow$

$x^2v = \frac{1}{5}x^5 + C \Rightarrow \underline{\underline{x^2/y = x^5/5 + C}}$ or $\underline{\underline{y = \frac{5x^2}{x^5 + C}}}$

$\boxed{5}$ Let $A(t)$ = Amt. (in grams) of dye in tank at time t (in minutes).

$$\frac{dA}{dt} = \binom{\text{rate}}{\text{in}} - \binom{\text{rate}}{\text{out}} = 0 - \left(\frac{A(t)\ g}{200\ L}\right)\left(\frac{2\ L}{\text{min.}}\right) \Rightarrow \frac{dA}{dt} = \frac{-A}{100} \Rightarrow \int \frac{1}{A}\,dA = \frac{-1}{100}\int dt \Rightarrow$$

$\ln A = -0.01t + C \Rightarrow A(t) = A_0 e^{-0.01t}$ (here A_0 replaces e^C).

Now, $A(0) = (200\ L)\left(\frac{1\ g}{L}\right) = 200\ g$ dye, which gives us $A_0 = 200$. So:

$A(t) = 200e^{-0.01t}$

Concentration function: $c(t) = \frac{\text{Amt. of dye at time } t}{\text{Volume of solution at } t} = \frac{A(t)}{200} = e^{-0.01t}$

Find when concentration is $0.01\,g/L$: $0.01 = e^{-0.01t} \Rightarrow -0.01t = \ln 0.01 \Rightarrow$

$t = \frac{\ln 0.01}{-0.01} = \underline{\underline{460.5 \text{ min.}}}$ ✓

$\boxed{6}$ $\frac{dA}{dt} = \binom{\text{rate}}{\text{in}} - \binom{\text{rate}}{\text{out}} = 0 - kA \Rightarrow \frac{dA}{dt} = -kA$, where k is a constant of proportionality.

$\int \frac{1}{A}\,dA = -k\int dt \Rightarrow \ln A = -kt + C \Rightarrow A(t) = A_0 e^{-kt}$

$A(0) = 450$ grams yields $A_0 = 450$, so $A(t) = 450e^{-kt}$

$A(24) = 100$ grams, so $100 = 450e^{-k(24)} \Rightarrow \frac{2}{9} = e^{-24k} \Rightarrow -24k = \ln 2/9 \Rightarrow k = \frac{\ln 2/9}{-24}$

$\Rightarrow k \approx 0.0627$

So $A(t) = 450e^{-0.0627t}$

Find t when $A(t) = 10$: $10 = 450e^{-0.0627t} \Rightarrow -0.0627t = \ln 1/45 \Rightarrow$

$t = \frac{\ln 1/45}{-0.0627} \approx \underline{\underline{60.71 \text{ hrs.}}}$ ✓