Math 250 Exam #1 Key (Fall 2008)

1. $\dfrac{dA}{dt} = kA^2$

2. Ordinary, nonlinear, second order, x independent, y dependent.

3a. $\dfrac{dx}{dt} = -\dfrac{4}{3}t$, so $f(t,x) = -\dfrac{4}{3}t$ and $\dfrac{\partial f}{\partial x}(x,y) = 0$. It is clear that f and $\dfrac{\partial f}{\partial x}$ are continuous everywhere. We can let our rectangle be

$$R = \{(x,y) : -\infty < x < \infty,\ -\infty < y < \infty\},$$

and since $(2, -\pi) \in R$ the Existence-Uniqueness Theorem implies that the initial-value problem (IVP) *does* have a unique solution.

3b. $\dfrac{dy}{dx} = -\dfrac{5x}{y}$, so $f(x,y) = -\dfrac{5x}{y}$ and $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{5x}{y^2}$. Note that the initial point $(1,0)$ is not in the domain of either f or $\partial f/\partial y$, and thus there exists no rectangle R containing $(1,0)$ on which both f and $\partial f/\partial y$ are continuous. As a result, the Existence-Uniqueness Theorem does *not* imply a unique solution.

4. We get $6\varphi''(x) - \varphi'(x) - 2\varphi(x) = 0 \quad\Rightarrow\quad 6m^2e^{mx} - me^{mx} - 2e^{mx} = 0 \quad\Rightarrow\quad 6m^2 - m - 2 = 0 \quad\Rightarrow$ $(3m-2)(2m+1) = 0$, and hence $m = -1/2, 2/3$.

5a. Separation of variables: $\dfrac{dy}{d\theta} = y\sin\theta \;\Rightarrow\; \dfrac{1}{y}\,dy = \sin\theta\,d\theta \;\Rightarrow\; \displaystyle\int \frac{1}{y}\,dy = \int \sin\theta\,d\theta \;\Rightarrow\; \ln|y| = -\cos\theta + C$. At the initial point we have $y = -3$, so we take $y < 0$ to get $\ln(-y) = -\cos\theta + C$. Now, substituting π for θ and -3 for y, we find that $\ln(3) = -\cos\pi + C$, and so $C = \ln 3 - 1$. The solution is $\ln(-y) = -\cos\theta + \ln 3 - 1$, or equivalently $y(x) = -3e^{-\cos\theta - 1}$.

5b. Separation of variables: $\displaystyle\int \frac{1}{y+1}\,dy = \int x^2\,dx \;\Rightarrow\; \ln|y+1| = \frac{1}{3}x^3 + C$. Since $y > 0$ at the initial point, we must have $y + 1 > 0$ as well, so $|y+1| = y+1$ and we obtain $\ln(y+1) = \dfrac{1}{3}x^3 + C$. Substituting 0 for x and 3 for y, we get $\ln(3+1) = 0 + C$ and hence $C = \ln 4$. The solution is $\ln(y+1) = \dfrac{1}{3}x^3 + \ln 4$, or equivalently $y(x) = 4e^{x^3/3} - 1$.

5c. Note that $t^3\dfrac{dx}{dt} + 3t^2x = \dfrac{d}{dt}\left(t^3x\right)$, so the equation becomes $\dfrac{d}{dt}\left(t^3x\right) = t$. Integrating with respect to t, we obtain $\displaystyle\int \frac{d}{dt}\left(t^3x\right)\,dt = \int t\,dt \;\Rightarrow\; t^3x = \frac{1}{2}t^2 + C$. Now, $x(2) = 0$ implies that $2^3 \cdot 0 = \frac{1}{2}\cdot 2^2 + C \;\Rightarrow\; C = -2$. Thus $t^3x = \dfrac{1}{2}t^2 - 2$, which we can solve for x to obtain the solution $x(t) = \dfrac{1}{2t} - \dfrac{2}{t^3}$.

6. $T(0) = 100$, $T(6) = 85$, $T(12) = 72$. Employ separation of variables to Newton's Law of Cooling: $\dfrac{dT}{dt} = k(M-T) \;\Rightarrow\; \displaystyle\int \frac{1}{M-T}\,dT = \int k\,dt \;\Rightarrow\; \ln|M-T| = kt + C$. Now, it's known that $M < T$

(i.e. the kitchen is cooler than the water), so we obtain

$$\ln(T - M) = kt + C. \tag{1}$$

From $T(0) = 100$ we get

$$C = \ln(100 - M), \tag{2}$$

from $T(6) = 85$ we get

$$\ln(85 - M) = 6k + C, \tag{3}$$

and from $T(12) = 72$ we get

$$\ln(72 - M) = 12k + C. \tag{4}$$

Using (2), substitute $\ln(100 - M)$ for C in (3) and (4) to obtain

$$\ln(85 - M) = 6k + \ln(100 - M) \quad \text{and} \quad \ln(72 - M = 12k + \ln(100 - M),$$

respectively. Thus

$$k = \frac{1}{6}\ln\left(\frac{85 - M}{100 - M}\right) \quad \text{and} \quad k = \frac{1}{12}\ln\left(\frac{72 - M}{100 - M}\right),$$

which yields the equation

$$\ln\left(\frac{72 - M}{100 - M}\right) = \ln\left(\frac{85 - M}{100 - M}\right)^2.$$

Then,

$$\frac{72 - M}{100 - M} = \left(\frac{85 - M}{100 - M}\right)^2 \;\Rightarrow\; (72 - M)(100 - M) = (85 - M)^2 \;\Rightarrow\; M = -12.5.$$

(Welcome to Hell's Kitchen!) From (2) we get $C = \ln(112.5)$. From (3) we get $\ln(97.5) = 6k + \ln(112.5)$, so $k = \frac{1}{6}\ln\left(\frac{13}{15}\right)$. Finally we return to (1) to obtain

$$\ln(T + 12.5) = \left(\ln \sqrt[6]{13/15}\right) t + \ln 112.5,$$

or equivalently

$$T(t) = -12.5 + 112.5\left(\frac{13}{15}\right)^{t/6}.$$