MATH 250 EXAM #2 (FALL 2007)

1 $\frac{\partial M}{\partial y} = xye^{xy} + e^{xy} + \frac{1}{y^2} = \frac{\partial N}{\partial x} \Rightarrow$ exact equation.

So, $\exists F \ni \frac{\partial F}{\partial x} = M$ & $\frac{\partial F}{\partial y} = N$, where $F(x,y) = A$ is the solution to the equation.

$\frac{\partial F}{\partial x} = M \Rightarrow dF = M\,dx \Rightarrow F = \int\left(ye^{xy} - \frac{1}{y}\right)dx = e^{xy} - \frac{x}{y} + g(y)$

Now, $\frac{\partial F}{\partial y} = N \Rightarrow xe^{xy} + \frac{x}{y^2} + g'(y) = xe^{xy} + \frac{x}{y^2} \Rightarrow g'(y) = 0 \Rightarrow g(y) = B$

Then $F(x,y) = e^{xy} - \frac{x}{y} + B$ & $F(x,y) = A \Rightarrow e^{xy} - \frac{x}{y} + B = A \Rightarrow \boxed{e^{xy} - \frac{x}{y} = C}$

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2 $M = 2xy$ & $N = y^2 - 3x^2$, so $\frac{\partial M}{\partial y} = 2x$ & $\frac{\partial N}{\partial x} = -6x$

Now, $\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{2x-(-6x)}{y^2-3x^2} = \frac{8x}{y^2-3x^2} \longrightarrow$ Not a function of x alone

But, $\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{-8x}{2xy} = -\frac{4}{y} \longrightarrow$ Is a function of y alone

So $\mu(y) = e^{\int -4/y\,dy} = e^{-4\ln y} = y^{-4}$ is an integrating factor. We get:

$$\underbrace{(2xy^{-3})}_{\hat{M}}dx + \underbrace{(y^{-2} - 3x^2y^{-4})}_{\hat{N}}dy = 0 \rightarrow \text{exact!}$$

$\exists F \ni \frac{\partial F}{\partial x} = \hat{M}$ & $\frac{\partial F}{\partial y} = \hat{N}$, where $F(x,y) = A$ is the solution to the equation.

$\frac{\partial F}{\partial x} = \hat{M} \Rightarrow dF = 2xy^{-3}dx \Rightarrow F = \int 2xy^{-3}dx = x^2y^{-3} + g(y)$

Now, $\frac{\partial F}{\partial y} = \hat{N} \Rightarrow -3x^2y^{-4} + g'(y) = y^{-2} - 3x^2y^{-4} \Rightarrow g'(y) = y^{-2} \Rightarrow g(y) = -y^{-1} + B$

Then $F(x,y) = x^2y^{-3} - \frac{1}{y} + B$ & $F(x,y) = A \Rightarrow x^2y^{-3} - \frac{1}{y} + B = A \Rightarrow \boxed{x^2y^{-3} - \frac{1}{y} = C}$

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3 $2xy\frac{dy}{dx} = -(x^2+y^2) \Rightarrow \frac{dy}{dx} = -\frac{x^2+y^2}{2xy} \Rightarrow \frac{dy}{dx} = -\frac{1+(y/x)^2}{2(y/x)}$, (1)

Let $v = \frac{y}{x}$, so $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, and then (1) becomes:

$v + x\frac{dv}{dx} = -\frac{1+v^2}{2v} \Rightarrow x\frac{dv}{dx} = -\frac{3v^2+1}{2v} \Rightarrow -\frac{2v}{3v^2+1}dv = \frac{1}{x}dx \Rightarrow$

$-2\int\frac{v}{3v^2+1}dv = \int\frac{1}{x}dx \Rightarrow -2\int\frac{1/6}{u}du = \ln|x| + A \Rightarrow -\frac{1}{3}\ln|u| = \ln|x| + A \Rightarrow$

$u = 3v^2+1$

$\ln\left(\frac{1}{\sqrt[3]{3v^2+1}}\right) = \underbrace{\ln|x| + A}_{\ln|x| + \ln B} \Rightarrow \frac{1}{\sqrt[3]{3(y/x)^2+1}} = |Bx| = \pm Bx = Cx,\ C \neq 0 \Rightarrow$

$\boxed{3xy^2 + x^3 = D}$ Also fine: $\boxed{\ln(3y^2/x^2+1)^{-1/3} = \ln|x| + A}$ or $\boxed{\frac{1}{\sqrt[3]{3xy^2+x^3}} = C}$

4 $\frac{dy}{dx} + y = -xy^3$, so let $v = y^{1-3} = y^{-2}$, whence $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$. Now...

$y^{-3}\frac{dy}{dx} + y^{-2} = -x \Rightarrow -\frac{1}{2}\frac{dv}{dx} + v = -x \Rightarrow \frac{dv}{dx} - 2v = 2x$, a linear eq.

Let $\mu(x) = e^{\int -2\,dx} = e^{-2x}$ be the integrating factor: $e^{-2x}\frac{dv}{dx} - 2ve^{-2x} = 2xe^{-2x} \Rightarrow$

$(e^{-2x}v)' = 2xe^{-2x} \Rightarrow e^{-2x}v = \int 2xe^{-2x}\,dx + C \Rightarrow v(x) = e^{2x}\left[2\int xe^{-2x}\,dx + C\right] \Rightarrow$

$v(x) = e^{2x}\left[2\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right) + C\right] = Ce^{2x} - x - \frac{1}{2} \Rightarrow \frac{1}{y^2} = Ce^{2x} - x - \frac{1}{2} \Rightarrow$

$y^2 = \frac{2}{Ce^{2x} - 2x - 1} \Rightarrow \boxed{Cy^2e^{2x} - 2xy^2 - y^2 = 2}$

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5 Let $x(t)$ = Mass of salt at time t, where $x(0) = 0$

$\frac{dx}{dt} = \left(\begin{matrix}\text{rate of}\\\text{salt input}\end{matrix}\right) - \left(\begin{matrix}\text{rate of}\\\text{salt output}\end{matrix}\right) = \left(\frac{0.25\text{ kg}}{1\text{ L}}\right)\left(\frac{5\text{ L}}{\text{min.}}\right) - \left(\frac{x\text{ kg}}{3t+90\text{ L}}\right)\left(\frac{2\text{ L}}{1\text{ min.}}\right) \Rightarrow$

$\frac{dx}{dt} = \frac{5}{4} - \frac{2x}{3t+90} \Rightarrow \frac{dx}{dt} + \frac{2}{3t+90}x = \frac{5}{4}$, a linear eq.

Let $\mu(x) = e^{\int \frac{2}{3t+90}dt} = e^{\frac{2}{3}\int \frac{1}{t+30}dt} = e^{\frac{2}{3}\ln(t+30)} = (t+30)^{2/3}$ be the integrating factor:

$(t+30)^{2/3}x' + \frac{2}{3}(t+30)^{-1/3}x = \frac{5}{4}(t+30)^{2/3} \Rightarrow \left[(t+30)^{2/3}x\right]' = \frac{5}{4}(t+30)^{2/3} \Rightarrow$

$(t+30)^{2/3}x = \frac{5}{4}\int (t+30)^{2/3}dt + C = \frac{5}{4}\cdot\frac{3}{5}(t+30)^{5/3} + C = \frac{3}{4}(t+30)^{5/3} + C \Rightarrow$

$x(t) = \frac{3}{4}(t+30) + C(t+30)^{-2/3}$.

From $x(0) = 0$ we get $0 = \frac{3}{4}\cdot 30 + C(0+30)^{-2/3} \Rightarrow 30^{-2/3}\cdot C = -\frac{90}{4} \Rightarrow C = 30^{2/3}\cdot -\frac{90}{4}$
$\Rightarrow C \approx -217.2$

So $\boxed{x(t) = 0.75(t+30) - 217.2(t+30)^{-2/3}}$

Note: concentration is given by $c(t) = \frac{x(t)}{V(t)} = \frac{x(t)}{3t+90} = 0.25 - 72.4(t+30)^{-5/3}$. As expected $\lim_{t\to\infty} c(t) = 0.25$ kg/L