Math 250 Exam #1 Key (Fall 2007)

1a. Ordinary, nonlinear, third order, x independent, y dependent.

1b. Partial, second order, r and t independent, N dependent.

2. We have $\dfrac{d}{dx}(y-\ln y)=2x \;\Rightarrow\; \dfrac{dy}{dx}-\dfrac{1}{y}\dfrac{dy}{dx}=2x \;\Rightarrow\; \dfrac{dy}{dx}=\dfrac{2x}{1-1/y} \;\Rightarrow\; \dfrac{dy}{dx}=\dfrac{2xy}{y-1}$, which is the given differential equation, so yes, it is an implicit solution.

3. Substituting $\varphi(x)$ for y, we get

$$\begin{aligned} x^2\varphi''(x)+7x\varphi'(x)+5\varphi(x)=0 \;&\Rightarrow\; x^2\cdot m(m-1)x^{m-2}+7x\cdot mx^{m-1}+5x^m=0\\ &\Rightarrow\; [m(m-1)+7m+5]x^m=0\\ &\Rightarrow\; m^2+6m+5=0 \;\Rightarrow\; (m+1)(m+5)=0, \end{aligned}$$

which gives $m=-5,-1$. Hence the functions $\varphi_1(x)=x^{-5}$ and $\varphi_2(x)=x^{-1}$ are particular solutions to the differential equation.

4. By separation of variables procedure we get $\displaystyle\int\frac{7r}{1-5r^2}dr=\int\frac{1}{x}dx$. Let $u=1-5r^2$, so by u-substitution procedure we get $du=-10r\,dr \;\Rightarrow\; -\frac{1}{10}du=r\,dr$; now, $\displaystyle\int\frac{-7/10}{u}\,du=\ln|x|+c \;\Rightarrow\; -\frac{7}{10}\ln|u|=\ln|x|+c \;\Rightarrow\; -\frac{7}{10}\ln|1-5r^2|=\ln|x|+c \;\Rightarrow\; 14\ln|1-5r^2|+20\ln|x|=c \;\Rightarrow\; \ln\left[x^{20}(1-5r^2)^{14}\right]=c$, where c is an arbitrary constant. This is a family of implicitly defined functions $r(x)$. We can also write $x^{20}(1-5r^2)^{14}=k$, where $k=e^c$ is again arbitrary, but it must be remembered that $k\neq 0$.

5. By separation of variables: $\displaystyle\int\frac{1}{2\sqrt{y+1}}\,dy=\int\sin x\,dx \;\Rightarrow\; \frac{1}{2}\cdot 2\sqrt{y+1}=-\cos x+c \;\Rightarrow\; \sqrt{y+1}+\cos x=c$. Now, using $y(\pi/2)=0$ we obtain $c=\sqrt{0+1}+\cos(\pi/2)=1$, so the solution is $\sqrt{y+1}+\cos x=1$, or more explicitly $y(x)=(1-\cos x)^2-1$.

6. Write $y'-\dfrac{1}{x}y=2x+1$. Let $\mu(x)=\exp\left(-\displaystyle\int\frac{1}{x}\,dx\right)=e^{-\ln x}=x^{-1}$, and multiply the differential equation to get $x^{-1}y'-x^{-2}y=2+x^{-1}$, whence $(x^{-1}y)'=2+x^{-1} \;\Rightarrow\; x^{-1}y=\displaystyle\int\left(2+\frac{1}{x}\right)dx \;\Rightarrow\; \frac{y}{x}=2x+\ln|x|+c$. Therefore the general solution is $y(x)=2x^2+x\ln|x|+cx$.

7. Following procedure, divide by t^3 to obtain $\dfrac{dx}{dt} + \dfrac{3}{t}x = \dfrac{1}{t^2}$. Now, $\mu(x) = \exp\left(\int \frac{3}{t}\,dt\right) = e^{3\ln t} = t^3$ is our integrating factor, which means (ironically) that we next multiply $\dfrac{dx}{dt} + \dfrac{3}{t}x = \dfrac{1}{t^2}$ by t^3 to obtain $t^3\dfrac{dx}{dt} + 3t^2x = t$ (the original equation). It may help to rewrite this as $t^3x' + 3t^2x = t$, which becomes $(t^3x)' = t$ and hence $t^3x = \displaystyle\int t\,dt = \frac{1}{2}t^2 + c$. From $x(2) = 0$ comes $2^3 \cdot 0 = \frac{1}{2} \cdot 2^2 + c$ and thus $c = -2$. So the solution is $t^3x = \dfrac{1}{2}t^2 - 2$, or more explicitly $x(t) = \dfrac{1}{2t} - \dfrac{2}{t^3}$.

8a. The solution curve corresponding to the initial condition $y(-2) = -2$ is depicted in red below.

8b. The solution curve corresponding to the initial condition $y(0) = 0$ is depicted in blue below.

8c. For the curve given by $y(-2) = -2$, it's seen that $y(x) \to -1^-$ as $x \to \infty$, and $y(x) \to -\infty$ as $x \to -\infty$. For the curve given by $y(0) = 0$, it's seen that $y(x) \to -1^+$ as $x \to \infty$, and $y(x) \to 1^-$ as $x \to -\infty$. (Incidentally the direction field happens to be that for the differential equation $y' = y^2 - 1$.)