Math 250 Exam #5 Key (Fall 2006)

1. Applying the substitution $u = t + 2$,

$$\begin{aligned}
\mathcal{L}[\sin(t+2)](s) &= \int_0^\infty e^{-st}\sin(t+2)\,dt = \int_2^\infty e^{-s(u-2)}\sin u\,du \\
&= \lim_{b\to\infty} e^{2s}\int_2^b e^{-su}\sin u\,du = \lim_{b\to\infty} e^{2s}\left[\frac{e^{-su}(-s\sin u - \cos u)}{s^2+1}\right]_2^\infty \\
&= \lim_{b\to\infty} e^{2s}\left[\frac{e^{-sb}(-s\sin b - \cos b)}{s^2+1} - \frac{e^{-2s}(-s\sin 2 - \cos 2)}{s^2+1}\right] \\
&= e^{2s}\left[0 - \frac{e^{-2s}(s\sin 2 + \cos 2)}{s^2+1}\right] = \frac{s\sin 2 + \cos 2}{s^2+1},
\end{aligned}$$

for all $s > 0$.

2. Here we use some algebra,

$$\begin{aligned}
\mathcal{L}[(t-2)^4](s) &= \mathcal{L}[t^4 - 8t^3 + 24t^2 - 32t + 16](s) \\
&= \mathcal{L}[t^4](s) - 8\mathcal{L}[t^3](s) + 24\mathcal{L}[t^2](s) - 32\mathcal{L}[t](s) + 16\mathcal{L}[1](s) \\
&= \frac{4!}{s^5} - 8\cdot\frac{3!}{s^4} + 24\cdot\frac{2!}{s^3} - 32\cdot\frac{1!}{s^2} + 16\cdot\frac{0!}{s^1} \\
&= \frac{24}{s^5} - \frac{48}{s^4} + \frac{48}{s^3} - \frac{32}{s^2} + \frac{16}{s}.
\end{aligned}$$

3. Use the product-to-sum identity $\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$:

$$\begin{aligned}
\mathcal{L}[\sin 2t \sin 5t](s) &= \mathcal{L}\left[\tfrac{1}{2}\left(\cos(2t-5t) - \cos(2t+5t)\right)\right](s) = \tfrac{1}{2}\mathcal{L}[\cos(-3t) - \cos(7t)](s) \\
&= \tfrac{1}{2}\mathcal{L}[\cos 3t](s)] - \tfrac{1}{2}\mathcal{L}[\cos 7t](s) = \frac{1}{2}\cdot\frac{s}{s^2+3^2} - \frac{1}{2}\cdot\frac{s}{s^2+7^2} \\
&= \frac{s}{2s^2+18} - \frac{s}{2s^2+98}
\end{aligned}$$

4. We have

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{2s+6}{s^2+4s+13}\right](t) &= \mathcal{L}^{-1}\left[\frac{2s+16}{(s+2)^2+3^2}\right](t) \\
&= 2\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right](t) + 4\mathcal{L}^{-1}\left[\frac{3}{(s+2)^2+3^2}\right](t) \\
&= 2e^{-2t}\cos 3t + 4e^{-2t}\sin 3t.
\end{aligned}$$

5. Solution to the IVP is $y(t) = 2e^{3t} + e^{2t}\sin t$.

6. Solution to the IVP is $y(t) = e^{-2t} - e^{-3t} + \left[\frac{7}{16} + \frac{1}{6}(t-2) - \frac{3}{4}e^{-2(t-2)} + \frac{5}{9}e^{-3(t-2)}\right]u(t-2)$.