MATH 250 EXAM #2 KEY (FALL 2006)

1) $\mu(x) = e^{\int 3/x\,dx} = e^{3\ln x} = x^3$.

Now ODE is: $x^3 y' + 3x^2 y = x^3(3x-2) \Rightarrow$
$(x^3 y)' = 3x^4 - 2x^3 \Rightarrow x^3 y = \int (3x^4 - 2x^3)\,dx + C$
$\Rightarrow y = x^{-3}\left[\frac{3}{5}x^5 - \frac{1}{2}x^4 + C\right] \Rightarrow$
$y = \frac{3}{5}x^2 - \frac{1}{2}x + x^{-3}C$.
From Initial Value: $1 = \frac{3}{5} - \frac{1}{2} + C \Rightarrow C = 9/10$
So, $y = \frac{3}{5}x^2 - \frac{1}{2}x + 0.9x^{-3}$ ✓

2) $\exists F \ni \frac{\partial F}{\partial x} = ye^{xy} - y^{-1}$ & $\frac{\partial F}{\partial y} = xe^{xy} + xy^{-2}$,

so $F = \int (ye^{xy} - y^{-1})\,dx + g(y) = e^{xy} - xy^{-1} + g(y)$

Now $N = \frac{\partial F}{\partial y} \Rightarrow xe^{xy} + xy^{-2} = xe^{xy} + xy^{-2} + g'(y)$

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$

Solution: $F(x,y) = C_2 \Rightarrow e^{xy} - xy^{-1} + C_1 = C_2 \Rightarrow$
$e^{xy} - xy^{-1} = C$

3) $\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{-6x - 2x}{2xy} = \frac{-8x}{2xy} = \frac{-4}{y}$

So $\mu(y) = e^{\int -4/y\,dy} = e^{-4\ln|y|} = e^{\ln y^{-4}} = y^{-4}$
Now ODE is: $2xy^{-3}\,dx + (y^{-2} - 3x^2y^{-4})\,dy = 0$
$\exists F \ni \frac{\partial F}{\partial x} = 2xy^{-3}$ & $\frac{\partial F}{\partial y} = y^{-2} - 3x^2y^{-4}$,
so $F = \int 2xy^{-3}\,dx + g(y) = x^2y^{-3} + g(y)$
Now $N = \partial F/\partial y \Rightarrow y^{-2} - 3x^2y^{-4} = -3x^2y^{-4} + g'(y)$
$\Rightarrow g'(y) = y^{-2} \Rightarrow g(y) = -y^{-1} + C_1$
Solution: $F(x,y) = C_2 \Rightarrow x^2y^{-3} - y^{-1} + C_1 = C_2$
$\Rightarrow x^2y^{-3} - y^{-1} = C$.

4) $\frac{dy}{d\theta} = \frac{\sec(y/\theta) + y/\theta}{1}$ $(\div \text{ by } \theta)$

Let $v = y/\theta$, so $y = v\theta \Rightarrow \frac{dy}{d\theta} = v + \theta\frac{dv}{d\theta}$, so
$v + \theta\frac{dv}{d\theta} = \sec v + v \Rightarrow \theta\frac{dv}{d\theta} = \sec v \Rightarrow$
$\cos v\,dv = \frac{1}{\theta}d\theta \Rightarrow \int \cos v\,dv = \int \frac{1}{\theta}d\theta \Rightarrow$
$\sin v = \ln|\theta| + C \Rightarrow \sin(y/\theta) = \ln|\theta| + C$

5) We get $y^2\frac{dy}{dx} + y^3 = e^x$
Let $v = y^3$, so $\frac{dv}{dx} = 3y^2\frac{dy}{dx} \Rightarrow y^2\frac{dy}{dx} = \frac{1}{3}\frac{dv}{dx}$
Then ODE becomes: $\frac{1}{3}\frac{dv}{dx} + v = e^x \Rightarrow$
$\frac{dv}{dx} + 3v = 3e^x$, a linear equation.

$\mu(x) = e^{\int 3\,dx} = e^{3x}$ is an integrating factor:
$e^{3x}v' + 3ve^{3x} = 3e^{4x} \Rightarrow$
$(ve^{3x})' = 3e^{4x} \Rightarrow ve^{3x} = 3\int e^{4x}\,dx$
$\Rightarrow ve^{3x} = \frac{3}{4}e^{4x} + C \Rightarrow$
$v = \frac{3}{4}e^x + Ce^{-3x} \Rightarrow y^3 = \frac{3}{4}e^x + Ce^{-3x}$

6) $\frac{\partial M}{\partial y} = \sec^2 y$, $\frac{\partial N}{\partial x} = \sec^2 y$
So ODE is exact. $\exists F \ni \frac{\partial F}{\partial x} = \tan y - 2$ and
$\frac{\partial F}{\partial y} = x\sec^2 y + 1/y$ ... so...
$F = \int N\,dy + h(x) = \int (x\sec^2 y + \frac{1}{y})\,dy + h(x) \Rightarrow$
$F = x\tan y + \ln y + h(x)$
$y = 1 > 0$ at initial value

So, $M = \frac{\partial F}{\partial x} \Rightarrow \tan y - 2 = \tan y + h'(x) \Rightarrow$
$h'(x) = -2 \Rightarrow h(x) = -2x + C_1$

Thus solution is $F(x,y) = C_2 \Rightarrow$
$x\tan y + \ln y - 2x + C_1 = C_2 \Rightarrow$
$x\tan y + \ln y - 2x = C$

Next, $y(0) = 1$ implies that
$0\cdot\tan(1) + \ln(1) - 2(0) = C \Rightarrow C = 0$

So $x\tan y + \ln y - 2x = 0$