Math 250 Exam #1 Key (Fall 2006)

1a. Ordinary, 2nd-order, x independent, y dependent, nonlinear.

1b. Ordinary, 4th-order, w independent, z dependent, linear.

2. Substitute $\varphi(x) = x^m$ for y in the differential equation to obtain:

$$\begin{aligned} 3x^2\varphi''(x) + 11x\varphi'(x) - 3\varphi(x) = 0 &\Rightarrow \\ 3m(m-1)x^2x^{m-2} + 11mxx^{m-1} - 3x^m = 0 &\Rightarrow \\ (3m^2 - 3m)x^m + 11mx^m - 3x^m = 0 &\Rightarrow \\ (3m^2 + 8m - 3)x^m = 0 &\Rightarrow \\ (3m-1)(m+3)x^m = 0. \end{aligned}$$

The last equation is satisfied for all $x \in \mathbb{R}$ if $m = 1/3$ or $m = -3$. Thus $\varphi(x) = x^{1/3}$ and $\varphi(x) = x^{-3}$ are both solutions to the original differential equation.

3. Substitution into $y'' + 4y = 5e^{-x}$ yields

$$\begin{aligned} (6\cos 2x - e^{-x})' + 4(3\sin 2x + e^{-x}) = 5e^{-x} &\Rightarrow \\ -12\sin 2x + e^{-x} + 12\sin 2x + 4e^{-x} = 5e^{-x} &\Rightarrow \\ 5e^{-x} = 5e^{-x}. \end{aligned}$$

Since we have obtained an identity, it follows that $y = 3\sin 2x + e^{-x}$ is indeed a solution to the differential equation.

4. We are given $(x_0, y_0) = (1, 0)$ and $h = 0.1$.

n	0	1	2	3	4
x_n	1.0	1.1	1.2	1.3	1.4
y_n	0.0000	0.1000	0.2090	0.3246	0.4441

5. From $x' = 3xt^2$ we obtain, by separation of variables, $\int \frac{1}{3x}\,dx = \int t^2\,dt \Rightarrow \frac{1}{3}\int \frac{1}{x}\,dx = \int t^2\,dt \Rightarrow \frac{1}{3}\ln|x| = \frac{1}{3}t^2 + c \Rightarrow \ln|x| = t^3 + c \Rightarrow e^{\ln|x|} = e^{t^3+c} \Rightarrow |x| = be^{t^3}$, where $b = e^c > 0$ is arbitrary. Now, $x(t) = \pm be^{t^3} = ae^{t^3}$, where $a = \pm b \neq 0$ is arbitrary. Observing that $x \equiv 0$ is also a solution to the original differential equation, we may permit $a = 0$ so that the solution is $x(t) = ce^{t^3}$ for c *any* arbitrary real number.

6. From $y' = 2x\cos^2 y$ we get $\frac{1}{\cos^2 y}y' = 2x$, and so by separation of variables comes $\int \sec^2 y\,dy = \int 2x\,dx$. Integrating gives $\int \tan y\,dy = x^2 + c$, where $c \in \mathbb{R}$ is arbitrary. From the initial condition it's seen that $y = \pi/4$ when $x = 0$, so $\tan \pi/4 = 0^2 + c \Rightarrow c = 1$. Therefore the solution to the initial value problem is $\tan y = x^2 + 1$, or more explicitly $y = \arctan(x^2 + 1)$. Note: the explicit solution is a curve for which $y \in (-\pi/2, \pi/2)$ for all $x \in (-\infty, \infty)$, which does indeed pass through the initial point $(0, \pi/4)$.