MATH 250 EXAM #3 KEY (FALL 2004)

1 $\mathcal{L}\{y''+9y\} = \mathcal{L}\{2\cos 3t\} \Rightarrow s^2Y(s) - sy(0) - y'(0) + 9Y(s) = 2\left(\frac{s}{s^2+9}\right) \Rightarrow$

$s^2Y(s) - s + 9Y(s) = \frac{2s}{s^2+9} \Rightarrow Y(s)(s^2+9) = \frac{2s}{s^2+9} + s \Rightarrow Y(s) = \frac{2s}{(s^2+9)^2} + \frac{s}{s^2+9}$

So $y(t) = 2\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{2(3)s}{(s^2+3^2)^2}\right\} + \cos 3t \Rightarrow$

$y(t) = \frac{1}{3}t\sin 3t + \cos 3t$ ✓

2 $f(t) = 3u(t-2) + (t^2-3)u(t-5)$ ✓

3 $\mathcal{L}\{y''+y\} = \mathcal{L}\{u(t-3)\} \Rightarrow s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s}e^{-3s} \Rightarrow$

$s^2Y(s) - 1 + Y(s) = \frac{1}{s}e^{-3s} \Rightarrow Y(s)(s^2+1) = \frac{1}{s}e^{-3s} + 1 \Rightarrow Y(s) = \frac{e^{-3s}}{s(s^2+1)} + \frac{1}{s^2+1}$

Decompose: $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow 1 = A(s^2+1) + (Bs+C)s \Rightarrow$

$1 = As^2 + A + Bs^2 + Cs \Rightarrow 1 = (A+B)s^2 + Cs + A \Rightarrow A = 1,\ C = 0,\ B = -1$

So $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$

Now $Y(s) = \frac{1}{s}e^{-3s} - \frac{s}{s^2+1}e^{-3s} + \frac{1}{s^2+1} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}e^{-3s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}e^{-3s}\right\} + \sin t \Rightarrow$

$y(t) = u(t-3) - \cos(t-3)u(t-3) + \sin t$ ✓

4 $\mathcal{L}\{y''-3y'+2y\} = \mathcal{L}\{\cos t\} \Rightarrow s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{s}{s^2+1} \Rightarrow$

$s^2Y(s) + 1 - 3sY(s) + 2Y(s) = \frac{s}{s^2+1} \Rightarrow Y(s)(s^2-3s+2) = \frac{s}{s^2+1} - 1 \Rightarrow$

$Y(s) = \frac{s}{(s^2+1)(s^2-3s+2)} - \frac{1}{s^2-3s+2} = \frac{-s^2+s-1}{(s^2+1)(s-2)(s-1)}$ ✓

5 $\frac{5s^2+34s+53}{(s+3)^2(s+1)} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s+1} \Rightarrow 5s^2+34s+53 = A(s+3)(s+1) + B(s+1) + C(s+3)^2 \Rightarrow$

$5s^2+34s+53 = As^2 + 4As + 3A + Bs + B + Cs^2 + 6Cs + 9C$

$= (A+C)s^2 + (4A+B+6C)s + (3A+B+9C)$

$$\begin{cases} A + C = 5, & (1) \\ 4A+B+6C = 34, & (2) \\ 3A+B+9C = 53, & (3) \end{cases}$$

(3) − (2) yields: $-A + 3C = 19$

Add to get: $4C = 24 \Rightarrow C = 6$

Then $A = -1$ & $B = 34 - 4A - 6C = 34 - 4(-1) - 6(6) = 2$

So $F(s) = -\frac{1}{s+3} + \frac{2}{(s+3)^2} + \frac{6}{s+1} \Rightarrow \mathcal{L}^{-1}\{F\} = -\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

So $\mathcal{L}^{-1}\{F\} = -e^{-3t} + 2e^{-3t}t + 6e^{-t} = (2t-1)e^{-3t} + 6e^{-t}$ ✓

6 Let $y(t) = z(t+1)$. Write differential equation using $t+1$ instead of t to get:

$z''(t+1) + 5z'(t+1) - 6z(t+1) = 21e^{(t+1)-1}$

Now IVP becomes:

$y''(t) + 5y'(t) - 6y(t) = 21e^t,\ y(0) = -1,\ y'(0) = 9$ ✓

$\boxed{7}$ We must find $\mathcal{L}^{-1}\left\{\ln\left(\frac{s-4}{s-3}\right)\right\}$, knowing that:

$$\mathcal{L}^{-1}\left\{\frac{d}{ds}\ln\left(\frac{s-4}{s-3}\right)\right\} = (-t)^1\mathcal{L}^{-1}\left\{\ln\left(\frac{s-4}{s-3}\right)\right\} \Rightarrow$$

$$-t\,\mathcal{L}^{-1}\left\{\ln\left(\frac{s-4}{s-3}\right)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-4)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-4} - \frac{1}{s-3}\right\} \Rightarrow$$

$$-t\,\mathcal{L}^{-1}\left\{\ln\left(\frac{s-4}{s-3}\right)\right\} = e^{4t} - e^{3t} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\ln\left(\frac{s-4}{s-3}\right)\right\} = \frac{e^{3t} - e^{4t}}{t} \quad ✓$$