MATH 250 EXAM #2 KEY (FALL 2004)

1 Characteristic equation: $2r^2+7r-4=0 \Rightarrow (2r-1)(r+4)=0 \Rightarrow r=\frac{1}{2}, -4$

So $u(t) = c_1 e^{t/2} + c_2 e^{-4t}$ ✓

2 Homogeneous solution: $y''+9y=0$ has char. eq. $r^2+9=0 \Rightarrow r = 0 \pm 3i \Rightarrow$

$y_h(t) = C_1 \cos 3t + C_2 \sin 3t$

Form for particular solution: $y_p(t) = t(A_3t^3+A_2t^2+A_1t+A_0)\cos 3t + t(B_3t^3+B_2t^2+B_1t+B_0)\sin 3t \Rightarrow$

$y_p(t) = (A_3t^4+A_2t^3+A_1t^2+A_0t)\cos 3t + (B_3t^4+B_2t^3+B_1t^2+B_0t)\sin 3t$ ✓

3 First look at the corresponding homogeneous equation: $y''-2y'+y=0$

Char. eq. is $r^2-2r+1=0 \Rightarrow (r-1)^2=0 \Rightarrow r=1,1$.

Homogeneous solution: $y_h(t) = C_1e^t + C_2te^t$

So particular solution will have the form $y_p(t) = At^2e^t$

Then $y_p'(t) = At^2e^t + 2Ate^t$ & $y_p''(t) = (At^2e^t + 2Ate^t) + (2Ate^t + 2Ae^t)$

So $y''_p - 2y'_p + y_p = 8e^t$

$(At^2e^t + 4Ate^t + 2Ae^t) - 2(At^2e^t + 2Ate^t) + (At^2e^t) = 8e^t \Rightarrow$

$\cancel{At^2e^t} + \cancel{4Ate^t} + 2Ae^t - \cancel{2At^2e^t} - \cancel{4Ate^t} + \cancel{At^2e^t} = 8e^t \Rightarrow$

$2Ae^t = 8e^t \Rightarrow A = 4$

So $y_p(t) = 4t^2e^t$ ✓

4 Homogeneous equation: $y''+y=0$. Characteristic equation: $r^2+1=0 \Rightarrow r=0\pm i$. Then we get

$y_h(t) = C_1\cos t + C_2 \sin t$ (homogeneous part of general solution, called homogeneous solution)

Particular solution to nonhomogeneous equation will have the form $y_p(t) = Ae^{-t}$. Then...

$y_p'(t) = -Ae^{-t}$ & $y_p''(t) = Ae^{-t}$

So $y_p'' + y_p = 2e^{-t} \Rightarrow Ae^{-t} + Ae^{-t} = 2e^{-t} \Rightarrow 2Ae^{-t} = 2e^{-t} \Rightarrow A = 1$. Then we get

$y_p(t) = e^{-t}$ (nonhomogeneous part of general solution, also called particular solution)

General solution: $y(t) = y_h(t) + y_p(t) = C_1 \cos t + C_2 \sin t + e^{-t}$

Then $y(0)=0 \Rightarrow 0 = C_1\cos(0) + C_2\sin(0) + e^0 \Rightarrow 0 = C_1 + 1 \Rightarrow \underline{C_1 = -1}$

So $y(t) = -\cos t + C_2 \sin t + e^{-t} \Rightarrow y'(t) = \sin t + C_2 \cos t - e^{-t}$

Then $y'(0) = 0 \Rightarrow 0 = \sin(0) + C_2\cos(0) - e^0 \Rightarrow 0 = C_2 - 1 \Rightarrow C_2 = 1$

$\therefore y(t) = \sin t - \cos t + e^{-t}$ ✓

5 See next page...

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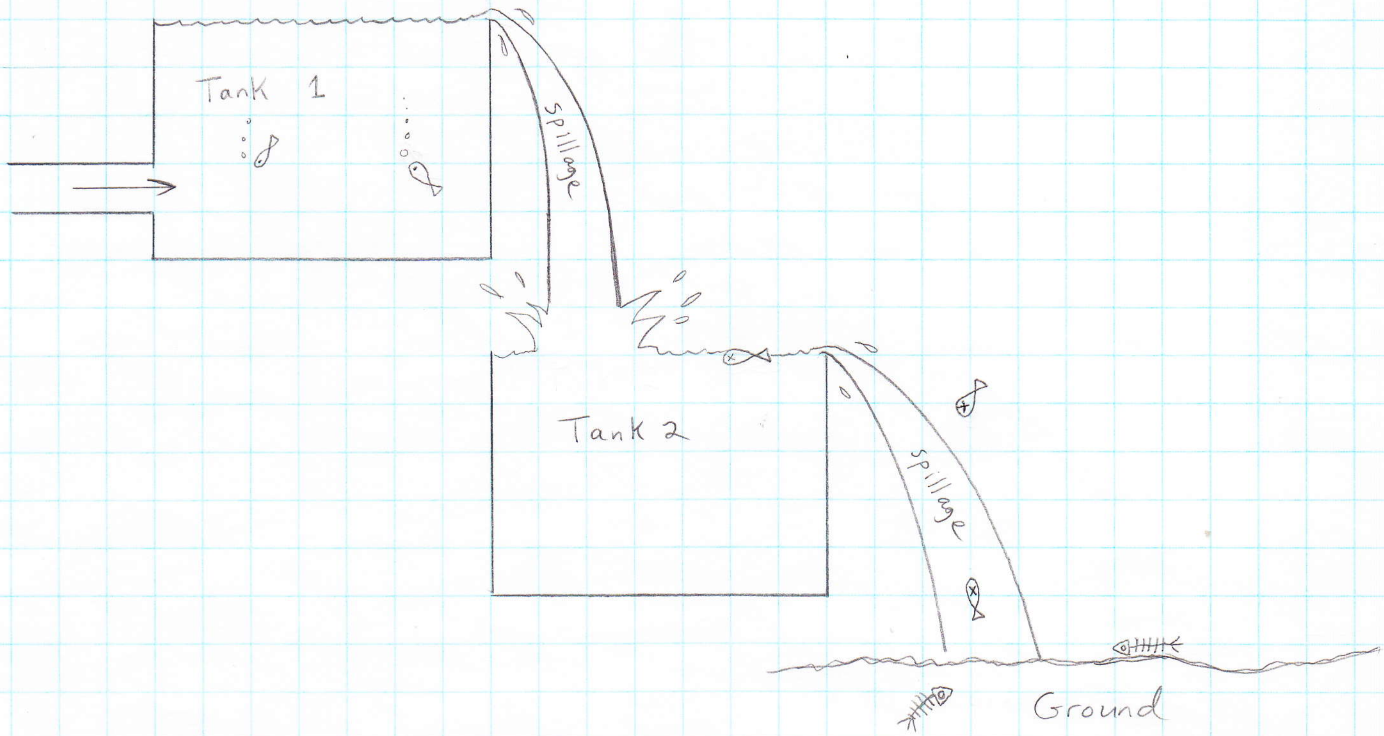

- Let $x(t)$ = Amount of salt in the 1st tank at time t. Then $\frac{x(t)}{60}$ is amount per gallon.
  $x(0) = x_0 > 0$ (starts with brine)
- Let $y(t)$ = Amount of salt in the 2nd tank at time t. Then $\frac{y(t)}{60}$ is amount per gallon.
  $y(0) = 0$ (starts with fresh water)
- First objective: Find the rate at which salt is entering the 2nd tank
- Final objective: Find the time t when $y(t)$ is a maximum.

$\frac{dx}{dt}$ = salt input − salt output $= 0 - \frac{3x}{60} = -\frac{x}{20} \Rightarrow \frac{1}{x}dx = -\frac{1}{20}dt \Rightarrow$
$\ln x = -\frac{1}{20}t + C_1 \Rightarrow x = C_2 e^{-t/20} \Rightarrow x(t) = x_0 e^{-t/20}$ (Amt. of salt in Tank 1 at time t)

So, salt is flowing into Tank 2 at rate of $\frac{3}{60}x(t) = \frac{3}{60}x_0 e^{-t/20} = \frac{x_0}{20}e^{-t/20}$

$\frac{dy}{dt}$ = salt input − salt output $= \frac{x_0}{20}e^{-t/20} - \frac{y}{20} \Rightarrow \frac{dy}{dt} + \frac{y}{20} = \frac{x_0}{20}e^{-t/20}$ (linear equation)

Let $\mu(t) = e^{\int P(t)dt} = e^{\int 1/20\, dt} = e^{t/20}$

Then $y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)Q(t)\,dt + C\right] = e^{-t/20}\left[\int e^{t/20}\cdot\frac{x_0}{20}e^{-t/20}\,dt + C\right] \Rightarrow$

$y(t) = e^{-t/20}\left[\frac{x_0}{20}t + C\right] = \frac{x_0}{20}te^{-t/20} + Ce^{-t/20}$

Then $y(0) = 0$ yields $C = 0$, so $y(t) = \frac{x_0}{20}te^{-t/20}$

Maximize $y(t)$: $y'(t) \overset{\text{set}}{=} 0 \Rightarrow \frac{x_0}{20}e^{-t/20}(1 - t/20) = 0 \Rightarrow t/20 = 1 \Rightarrow \underline{t = 20 \text{ minutes}}$

6 $m = 2$, $k = 40$, $b = 8\sqrt{5}$, $y(0) = 0.1$, $y'(0) = 2$

Then $2y'' + 8\sqrt{5}\,y' + 40y = 0 \Rightarrow y'' + 4\sqrt{5}\,y' + 20y = 0 \Rightarrow r^2 + 4\sqrt{5}\,r + 20 = 0 \Rightarrow$

$r = \dfrac{-4\sqrt{5} \pm \sqrt{80 - 4(1)(20)}}{2} = -2\sqrt{5} \pm 0 = -2\sqrt{5}$

So $y(t) = c_1 e^{-2\sqrt{5}t} + c_2 t e^{-2\sqrt{5}t}$

Since $y(0) = 0.1$, $0.1 = c_1$ & $y(t) = 0.1e^{-2\sqrt{5}t} + c_2 t e^{-2\sqrt{5}t}$

$y'(t) = -0.2\sqrt{5}\,e^{-2\sqrt{5}t} - 2\sqrt{5}\,c_2 t e^{-2\sqrt{5}t} + c_2 e^{-2\sqrt{5}t}$

Since $y'(0) = 2$, $2 = -0.2\sqrt{5} + c_2 \Rightarrow c_2 = 2 + 0.2\sqrt{5}$

Then $y'(t) = -0.2\sqrt{5}\,e^{-2\sqrt{5}t} - 2\sqrt{5}(2 + 0.2\sqrt{5})te^{-2\sqrt{5}t} + (2 + 0.2\sqrt{5})e^{-2\sqrt{5}t} \Rightarrow$

Set $y'(t) = 0$: $-0.\cancel{2\sqrt{5}} - 2\sqrt{5}(2 + 0.2\sqrt{5})t + (2 + 0.\cancel{2\sqrt{5}}) = 0 \Rightarrow$

$-4\sqrt{5}\,t - 2t + 2 = 0 \Rightarrow (4\sqrt{5} + 2)t = 2 \Rightarrow t = \dfrac{2}{4\sqrt{5} + 2}$ (time of maximum displacement)

So $y(t) = 0.1e^{-2\sqrt{5}t} + (2 + 0.2\sqrt{5})te^{-2\sqrt{5}t}$ is maximized when $t \approx 0.1827$ sec.

Maximum displacement $= y(0.1827) = 0.2417$ meter

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7 $m = 5$, $v_0 = 50$, $g = 9.81$, $b = 10$

Equation given: $v(t) = \dfrac{mg}{b} + \left(v_0 - \dfrac{mg}{b}\right)e^{-bt/m}$, the velocity of the object at time t.

Let "down" be the positive direction.

$x(t) \equiv$ displacement of object at time t, where $x(0) = 0$ and we must find the time when $x(t) = 500$

Now, $x(t) = \int v(t)\,dt = \int \frac{mg}{b}\,dt + \left(v_0 - \frac{mg}{b}\right)\int e^{-bt/m}\,dt$

$= \dfrac{mg}{b}t + \left(v_0 - \dfrac{mg}{b}\right)\left(-\dfrac{m}{b}e^{-bt/m}\right) + C$

Since $x(0) = 0$, we get $\left(v_0 - \frac{mg}{b}\right)\left(-\frac{m}{b}\right) + C = 0 \Rightarrow C = \frac{m}{b}\left(v_0 - \frac{mg}{b}\right)$

So $x(t) = \dfrac{mg}{b}t + \dfrac{m}{b}\left(v_0 - \dfrac{mg}{b}\right)\left(1 - e^{-bt/m}\right) \Rightarrow \underline{x(t) = 4.905t + 22.5475\left(1 - e^{-2t}\right)}$ ✓

So $x(t) = 500 \Rightarrow 4.905t + 22.5475 - 22.5475e^{-2t} = 500 \Rightarrow$

$4.905t - 22.5475e^{-2t} = 477.4525 \Rightarrow$

$t - 4.5968e^{-2t} = 97.3400$, (1)

Imagine that $t = 97.34$. Then (1) becomes: $97.34 - \underbrace{4.5968e^{-194.68}}_{\text{Practically } 0} = 97.34$, and we see the equation is satisfied. Thus $\underline{t \approx 97.34 \text{ seconds}}$ ✓