MATH 250
FALL 2021
EXAM 3

NAME:

1. [20 pts.] Find the general solution by the Method of Undetermined Coefficients:

$$y'' + y' = x + \sin 2x.$$

2. [20 pts.] Find the general solution by the Method of Variation of Parameters:

$$y'' + y = \sec^3 x.$$

3. [20 pts.] Solve the system of differential equations by elimination:

$$\begin{cases} (D+1)x + (D-1)y = 2 \\ \qquad\quad 3x + (D+2)y = -1 \end{cases}$$

4. [20 pts.] Solve the nonlinear differential equation

$$x^2 y'' + (y')^2 = 0$$

5. [10 pts.] Obtain the first four nonzero terms of a Taylor series solution (centered at 0) to the initial-value problem

$$y'' - 2y^2 = 4x, \quad y(0) = -1, \quad y'(0) = 2.$$

Method of Undetermined Coefficients. Let $P_m(x)$ be a nonzero polynomial of degree m, and let $y_p(x)$ denote a particular solution to $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x)$.

1. If $f(x) = P_m(x)e^{\alpha x}$, then

$$y_p(x) = x^s e^{\alpha x} \sum_{k=0}^{m} A_k x^k,$$

where $s = 0$ if α is not a root of the auxiliary equation, otherwise s equals the multiplicity of α as a root of the auxiliary equation.

2. If $f(x) = P_m(x)e^{\alpha x}\cos\beta x$ or $f(x) = P_m(x)e^{\alpha x}\sin\beta x$ for $\beta \neq 0$, then

$$y_p(x) = x^s e^{\alpha x} \left(\cos\beta x \sum_{k=0}^{m} A_k x^k + \sin\beta x \sum_{k=0}^{m} B_k x^k \right),$$

where $s = 0$ if $\alpha + i\beta$ is not a root of the auxiliary equation, otherwise s equals the multiplicity of $\alpha + i\beta$ as a root of the auxiliary equation.

Method of Variation of Parameters for $\mathbf{y'' + p_1(x)y' + p_0(x)y = q(x)}$:

$$u_1(x) = -\int \frac{y_2(x)q(x)}{\mathcal{W}[y_1, y_2](x)}\,dx \quad \text{and} \quad u_2(x) = \int \frac{y_1(x)q(x)}{\mathcal{W}[y_1, y_2](x)}\,dx$$

Some Most Excellent Formulae.

1. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$, for $a \in (0, \infty)$
2. $\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$, for $a \neq 0$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + c$, for $a \in (0, \infty)$
4. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + c = \ln|\sec x| + c$
5. $\displaystyle\int \cot x\,dx = \ln|\sin x| + c$
6. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + c$
7. $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + c$