7

THE LAPLACE TRANSFORM

7.1 – IMPROPER RIEMANN INTEGRALS

In this section we undertake a study of improper integrals. Simply put, an improper Riemann integral is any sort of “integral” that does not conform to the definition of a Riemann integral, which requires an interval of integration \([a, b]\) that is closed and bounded, and also a bounded real-valued function \(f\) that is defined at every point in the interval of integration so that \([a, b] \subseteq \text{Dom}(f)\). If \(f : [0, 2] \to \mathbb{R}\) is given by

\[
f(x) = \begin{cases} 
  x^3, & \text{if } 0 \leq x < 2 \\
  10, & \text{if } x = 2
\end{cases}
\]

then the integral \(\int_0^2 f\) is a completely “proper” Riemann integral which can be evaluated using the definition, even though \(f\) has a discontinuity at 2. This is therefore not the kind of integral we’re concerned about at present.

Given a closed, bounded interval \([a, b]\), the symbol \(\mathcal{R}[a, b]\) denotes the set of all functions that are Riemann integrable on \([a, b]\). Suppose \(f \in [a, t]\) for all \(t \geq a\); that is, \(f\) is a function that is integrable on \([a, t]\) for all \(t \in \mathbb{R}\) such that \(t \geq a\). This means that

\[
\int_a^t f \in \mathbb{R}
\]

for each \(t \geq a\). This observation leads us to ask whether \(\int_a^t f\) tends to some limiting value \(L \in \mathbb{R}\) as \(t \to \infty\); that is, does the limit

\[
\lim_{x \to \infty} \int_a^t f
\]

exist in \(\mathbb{R}\)? Such a question arises frequently in applications, and so motivates the following definition.

**Definition 7.1.** If \(f \in \mathcal{R}[a, b]\) for all \(b \geq a\), then we define

\[
\int_a^\infty f = \lim_{b \to \infty} \int_a^b f
\]

and say that \(\int_a^\infty f\) converges to \(L\) if \(\int_a^\infty f = L\) for some \(L \in \mathbb{R}\). Otherwise we say that \(\int_a^\infty f\) diverges.
If \( f \in \mathcal{R}[a, b] \) for all \( a \leq b \), then we define
\[
\int_{-\infty}^{b} f = \lim_{a \to -\infty} \int_{a}^{b} f
\]
and say that \( \int_{-\infty}^{b} f \) converges to \( L \) if \( \int_{-\infty}^{b} f = L \) for some \( L \in \mathbb{R} \). Otherwise we say that \( \int_{-\infty}^{b} f \) diverges.

If an improper integral converges to some real number \( L \) then it is customary to say simply that the integral “converges” or is “convergent.” An integral that “diverges” is also said to be “divergent.” Any integral, such as those defined above, that has an unbounded interval of integration is called an improper integrals of the first kind.

Example 7.2. Determine whether
\[
\int_{1}^{\infty} \frac{\ln(x)}{x^2} \, dx
\]
converges or diverges. Evaluate if convergent.

Solution. It will be easier to first determine the indefinite integral
\[
\int \frac{\ln(x)}{x^2} \, dx.
\]
We start with a substitution: let \( w = \ln(x) \), so that \( dw = (1/x) \, dx \) and \( e^w = e^{\ln(x)} = x \); now,
\[
\int \frac{\ln(x)}{x^2} \, dx = \int w e^{-w} \, dw.
\]
Next, we employ integration by parts, letting \( u'(w) = e^{-w} \) and \( v(w) = w \) to obtain
\[
\int w e^{-w} \, dw = -we^{-w} + \int e^{-w} \, dw = -we^{-w} - e^{-w} + C.
\]
Hence,
\[
\int \frac{\ln(x)}{x^2} \, dx = -\ln(x) \cdot \frac{1}{x} - \frac{1}{x} + C = -\frac{\ln(x) + 1}{x} + C.
\]
Now we turn to the improper integral,
\[
\int_{1}^{\infty} \frac{\ln(x)}{x^2} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{\ln(x)}{x^2} \, dx = \lim_{b \to \infty} \left[ -\frac{\ln(x) + 1}{x} \right]_{1}^{b}
\]
\[
= \lim_{b \to \infty} \left[ -\frac{\ln(b) + 1}{b} + \frac{\ln(1) + 1}{1} \right] = \lim_{b \to \infty} \left( \frac{b - \ln(b) + 1}{b} \right)
\]
\[
= \lim_{b \to \infty} \left( 1 - \frac{1}{b} \right) = 1,
\]
using L’Hôpital’s Rule for the penultimate equality.
Therefore the improper integral is convergent, and its value is 1. \( \blacksquare \)
Given an improper integral such as \( \int_a^\infty f \), if \( f(x) \geq 0 \) for all \( x \in [a, \infty) \), then the value of the integral can be naturally interpreted as being the area under the curve \( y = f(x) \) for \( x \geq a \). If the integral is divergent (in this case it will equal \( \infty \)), then the area is said to be infinite; and if the integral is convergent, then the area is set equal to the real number the integral converges to. Thus the area under the curve \( y = \ln(x)/x^2 \), illustrated in Figure 1, is considered to be 1. Thus, the shaded region has an infinite “perimeter” and yet a finite area!

**Proposition 7.3.** Suppose that \( f \in R[s,t] \) for all \( -\infty < s < t < \infty \). If \( \int_{-\infty}^c f \) and \( \int_c^\infty f \) converge for some \( c \in \mathbb{R} \), then for any \( \hat{c} \neq c \) the integrals \( \int_{-\infty}^{\hat{c}} f \) and \( \int_{\hat{c}}^\infty f \) also converge, and

\[
\int_{-\infty}^{\hat{c}} f + \int_{\hat{c}}^\infty f = \int_{-\infty}^{c} f + \int_{c}^\infty f
\]

**Proof.** Suppose \( \int_{-\infty}^{c} f \) and \( \int_{c}^{\infty} f \) converge for some \( c \in \mathbb{R} \), meaning the limits

\[
\lim_{a \to -\infty} \int_{a}^{c} f \quad \text{and} \quad \lim_{b \to \infty} \int_{c}^{b} f
\]

both exist. Let \( \hat{c} < c \).

For all \( b > c \) we have

\[
\int_{\hat{c}}^{b} f = \int_{\hat{c}}^{c} f + \int_{c}^{b} f,
\]

where \( \int_{\hat{c}}^{c} f, \int_{c}^{b} f \in \mathbb{R} \) since \( f \) is integrable on \([\hat{c}, c]\) and \([c, b]\), and so

\[
\int_{\hat{c}}^{\infty} f = \lim_{b \to \infty} \int_{\hat{c}}^{b} f = \lim_{b \to \infty} \left( \int_{\hat{c}}^{c} f + \int_{c}^{b} f \right) = \int_{\hat{c}}^{c} f + \lim_{b \to \infty} \int_{c}^{b} f = \int_{\hat{c}}^{c} f + \int_{c}^{\infty} f. \quad (1)
\]

Observing that \( \int_{\hat{c}}^{c} f, \int_{c}^{\infty} f \in \mathbb{R} \), we readily conclude that \( \int_{\hat{c}}^{\infty} f \in \mathbb{R} \) and hence \( \int_{\hat{c}}^{c} f \) converges. For all \( a < \hat{c} \) we have

\[
\int_{a}^{\hat{c}} f = \int_{a}^{c} f - \int_{c}^{\hat{c}} f,
\]

where \( \int_{a}^{c} f, \int_{c}^{\hat{c}} f \in \mathbb{R} \) since \( f \) is integrable on \([a, c]\) and \([c, \hat{c}]\), and so

\[
\int_{-\infty}^{\hat{c}} f = \lim_{a \to -\infty} \int_{a}^{\hat{c}} f = \lim_{a \to -\infty} \left( \int_{a}^{c} f - \int_{c}^{\hat{c}} f \right) = \lim_{a \to -\infty} \int_{a}^{c} f - \lim_{a \to -\infty} \int_{c}^{\hat{c}} f = \int_{-\infty}^{c} f - \int_{c}^{\hat{c}} f. \quad (2)
\]

Observing that \( \int_{-\infty}^{c} f, \int_{c}^{\hat{c}} f \in \mathbb{R} \), we readily conclude that \( \int_{-\infty}^{\hat{c}} f \in \mathbb{R} \) and hence \( \int_{-\infty}^{\hat{c}} f \) converges.

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**FIGURE 1.** The area under the curve \( y = \ln(x)/x^2 \).
Finally, combining (1) and (2), we obtain
\[ \int_{-\infty}^{c} f + \int_{c}^{\infty} f = \left( \int_{-\infty}^{c} f - \int_{c}^{\infty} f \right) + \left( \int_{c}^{c} f + \int_{c}^{\infty} f \right) = \int_{-\infty}^{c} f + \int_{c}^{\infty} f, \]
as desired. ■

Due to Proposition 7.3 we can unambiguously define an improper integral of the first kind whose interval of integration is \((-\infty, \infty)\).

**Definition 7.4.** Suppose that \(f \in \mathcal{R}[a,b]\) for all \(-\infty < a < b < \infty\). If \(\int_{-\infty}^{c} f\) and \(\int_{c}^{\infty} f\) both converge for some \(-\infty < c < \infty\), then we define
\[ \int_{-\infty}^{\infty} f = \int_{-\infty}^{c} f + \int_{c}^{\infty} f. \]
and say that \(\int_{-\infty}^{\infty} f\) converges. Otherwise we say \(\int_{-\infty}^{\infty} f\) diverges.

It should be stressed that \(\int_{-\infty}^{\infty} f\) can not be reliably evaluated simply by computing the limit
\[ \lim_{b \to \infty} \int_{-b}^{b} f, \]
as the next example illustrates.

**Example 7.5.** Show that
\[ \int_{-\infty}^{\infty} \frac{2x}{1 + x^2} \, dx \]
diverges, and yet
\[ \lim_{b \to \infty} \int_{-b}^{b} \frac{2x}{1 + x^2} \, dx = 0. \]

**Solution.** Letting \(u = 1 + x^2\) gives \(du = 2x \, dx\). Then
\[ \int_{0}^{b} \frac{2x}{1 + x^2} \, dx = \int_{1}^{1+b^2} \frac{1}{u} \, du = [\ln |u|]_{1}^{1+b^2} = \ln(1 + b^2) - \ln(1) = \ln(1 + b^2), \]
and so
\[ \int_{0}^{\infty} \frac{2x}{1 + x^2} \, dx = \lim_{b \to \infty} \int_{0}^{b} \frac{2x}{1 + x^2} \, dx = \lim_{b \to \infty} \ln(1 + b^2) = \infty. \]
Thus
\[ \int_{0}^{\infty} \frac{2x}{1 + x^2} \, dx \]
diverges, and therefore
\[ \int_{-\infty}^{\infty} \frac{2x}{1 + x^2} \, dx \]
diverges as well.

On the other hand, again employing the substitution \(u = 1 + x^2\) we find that
\[ \int_{-b}^{b} \frac{2x}{1 + x^2} \, dx = \int_{1+b^2}^{1+1} \frac{1}{u} \, du = 0, \]
and so
\[
\lim_{b \to \infty} \int_{-b}^{b} \frac{2x}{1 + x^2} \, dx = \lim_{b \to \infty} (0) = 0. 
\]

An **improper integral of the second kind** is an integral of the form
\[
\int_{a}^{b} f, 
\]
where \(-\infty < a < b < \infty\), for which there exists some \(p \in [a, b]\) such that \(p \notin \text{Dom}(f)\). The following definition establishes how such an integral is to be evaluated, if it can be evaluated at all, in the case when \(p = a\) or \(p = b\).

**Definition 7.6.** If \(f \in \mathcal{R}[c, b]\) for all \(c \in (a, b]\) and \(a \notin \text{Dom}(f)\), then we define
\[
\int_{a}^{b} f = \lim_{c \to a^+} \int_{c}^{b} f 
\]
and say that \(\int_{a}^{b} f\) **converges to** \(L\) if \(\int_{a}^{b} f = L \) for some \(L \in \mathbb{R}\). Otherwise we say that \(\int_{a}^{b} f\) **diverges**.

If \(f \in \mathcal{R}[a, c]\) for all \(c \in [a, b)\) and \(b \notin \text{Dom}(f)\), then we define
\[
\int_{a}^{b} f = \lim_{c \to b^-} \int_{a}^{c} f 
\]
and say that \(\int_{a}^{b} f\) **converges to** \(L\) if \(\int_{a}^{b} f = L \) for some \(L \in \mathbb{R}\). Otherwise we say that \(\int_{a}^{b} f\) **diverges**.

Very often if \(f\) is continuous on, say, \((a, b]\) and \(a \notin \text{Dom}(f)\), then \(f\) has a vertical asymptote at \(a\); that is, \(\lim_{x \to a^+} f(x) = \pm \infty\). However, it could just be that a value for \(f\) is simply not specified at \(a\) by construction. For example for the function
\[
\varphi(x) = \begin{cases} 
3x^2, & \text{if } x < 5 \\
4 - 8x, & \text{if } x > 5
\end{cases}
\]
it’s seen that \(\varphi(5)\) is left undefined, and so the integral \(\int_{5}^{9} \varphi\) is an improper integral of the second kind. By Definition 7.6 we obtain
\[
\int_{5}^{9} \varphi = \lim_{c \to 5^+} \int_{c}^{9} (4 - 8x) \, dx = \lim_{c \to 5^+} \left[ 4x - 4x^2 \right]_{c}^{9} = \lim_{c \to 5^+} \left[ (4(9) - 4(9)^2) - (4c - 4c^2) \right] \\
= (4(9) - 4(9)^2) - (4(5) - 4(5)^2) = -208, 
\]
which shows that \(\int_{5}^{9} \varphi\) is convergent.

**Example 7.7.** Determine whether
\[
\int_{-1}^{9} \frac{1}{x^2} \, dx
\]
converges or diverges. Evaluate if convergent.
Solution. The function \( f(x) = 1/x^2 \) being integrated has a vertical asymptote at \( x = 0 \), which is the right endpoint of the interval of integration \([-1, 0]\). By Definition 7.6 we obtain
\[
\int_{-1}^{0} \frac{1}{x^2} \, dx = \lim_{c \to 0^-} \int_{-1}^{c} \frac{1}{x^2} \, dx = \lim_{c \to 0^-} \left[ \frac{-1}{x} \right]_{-1}^{c} = \lim_{c \to 0^-} \left( \frac{-1}{c} - 1 \right) = \infty,
\]
which shows that the improper integral is divergent.

Example 7.8. Determine whether
\[
\int_{0}^{2} \frac{x}{\sqrt{4-x^2}} \, dx
\]
converges or diverges. Evaluate if convergent.

Solution. Here \( x/\sqrt{4-x^2} \) has a vertical asymptote at \( x = 2 \), the right endpoint of the interval of integration \([0, 2]\). By Definition 7.6
\[
\int_{0}^{2} \frac{x}{\sqrt{4-x^2}} \, dx = \lim_{c \to 2^-} \int_{0}^{c} \frac{x}{\sqrt{4-x^2}} \, dx,
\]
and so, letting \( u = 4 - x^2 \) so that \( x \, dx = -\frac{1}{2} du \), we obtain
\[
\lim_{c \to 2^-} \int_{0}^{c} \frac{x}{\sqrt{4-x^2}} \, dx = \lim_{c \to 2^-} \int_{4}^{4-c^2} \frac{-1/2}{\sqrt{u}} \, du = \lim_{c \to 2^-} \left( -\frac{1}{2} \left[ 2\sqrt{u} \right]_{4-c^2}^{4} \right)
\]
\[
= \lim_{c \to 2^-} \left( 2 - \sqrt{4-c^2} \right) = 2 - \sqrt{4-2^2} = 2.
\]
Hence the improper integral is convergent, and its value is 2.

The next definition addresses the circumstance when a function \( f \) is not defined at some point \( p \) in the interior of an interval of integration. Again, this is commonly due to \( f \) having a vertical asymptote at \( p \), so that
\[
\lim_{x \to p^+} |f(x)| = \infty \quad \text{or} \quad \lim_{x \to p^-} |f(x)| = \infty,
\]
but other scenarios are possible.

Definition 7.9. Suppose that \( f \in \mathcal{R}[a,c] \) for all \( c \in [a,p) \), \( f \in \mathcal{R}[c,b] \) for all \( c \in (p,b] \), and \( p \notin \text{Dom}(f) \). If \( \int_{a}^{p} f \) and \( \int_{p}^{b} f \) both converge, then we define
\[
\int_{a}^{b} f = \int_{a}^{p} f + \int_{p}^{b} f.
\]
and say that \( \int_{a}^{b} f \) converges. Otherwise we say \( \int_{a}^{b} f \) diverges.

Example 7.10. Determine whether
\[
\int_{-2}^{3} \frac{1}{x^4} \, dx
\]
converges or diverges. Evaluate if convergent.
Solution. Here $1/x^4$ has a vertical asymptote at $x = 0$, an interior point of the interval of integration $[-2, 3]$. Now, by Definition 7.6
\[
\int_0^3 \frac{1}{x^4} \, dx = \lim_{c \to 0^+} \int_c^3 \frac{1}{x^4} \, dx = \lim_{c \to 0^+} \left[ -\frac{1}{x^3} \right]_c^3 = \lim_{c \to 0^+} \left( -\frac{1}{27} + \frac{1}{c^3} \right) = \infty,
\]
which shows that $\int_0^3 x^{-4} \, dx$ is divergent. Thus, since
\[
\int_0^3 x^{-4} \, dx \quad \text{and} \quad \int_{-2}^0 x^{-4} \, dx
\]
cannot both be convergent, by Definition 7.9 it’s concluded that $\int_{-2}^3 x^{-4} \, dx$ is divergent. ■

The integral treated in Example 7.10 like all improper integrals of the second kind, does not look improper at first glance. If one is careless and undertakes to evaluate the integral by conventional means, one is likely to arrive at a reasonable-looking answer without ever suspecting that something is amiss:
\[
\int_{-2}^3 \frac{1}{x^4} \, dx = \left[ -\frac{1}{x^3} \right]_{-2}^3 = -\frac{1}{27} + \frac{1}{-8} = -\frac{35}{216},
\]
which is incorrect! So, before attempting to evaluate a definite integral, it is necessary to check that the integral is not improper in some way.

It is possible to have an integral that is improper in more than one sense, such as
\[
\int_0^\infty \frac{1}{x^2} \, dx.
\]
Here we have an integral of $f$ over an unbounded interval $[0, \infty)$, so it’s an improper integral of the first kind, and also $f$ is undefined at 0, so it’s an improper integral of the second kind. Such an integral is called a mixed improper integral.

Definition 7.11. If $f \in \mathcal{R}[s, t]$ for all $a < s < t < \infty$, $a \not\in \text{Dom}(f)$, and $\int_a^c f$ and $\int_c^\infty f$ both converge for some $c \in (a, \infty)$, then we define
\[
\int_a^\infty f = \int_a^c f + \int_c^\infty f
\]
and say $\int_a^\infty f$ converges. Otherwise we say $\int_a^\infty f$ diverges.

If $f \in \mathcal{R}[s, t]$ for all $-\infty < s < t < b$, $b \not\in \text{Dom}(f)$, and $\int_c^{-\infty} f$ and $\int_c^b f$ both converge for some $c \in (-\infty, b)$, then we define
\[
\int_{-\infty}^b f = \int_{-\infty}^c f + \int_c^b f
\]
and say $\int_{-\infty}^b f$ converges. Otherwise we say $\int_{-\infty}^b f$ diverges.

Example 7.12. Determine whether the mixed improper integral
\[
\int_0^\infty \frac{1}{\sqrt{x(1+x)}} \, dx
\]
converges or diverges. Evaluate if convergent.
Solution. We start by determining the indefinite integral
\[ \int \frac{1}{\sqrt{x(1 + x)}} \, dx. \]
Let \( u = \sqrt{x} \), so that \( 1 + u^2 = 1 + x \) and we replace \( dx \) with \( 2u \, du \) to obtain
\[ \int \frac{1}{\sqrt{x(1 + x)}} \, dx = \int \frac{2u}{u(u^2 + 1)} \, du = 2 \int \frac{1}{u^2 + 1} \, du \]
\[ = 2 \arctan(u) + c = 2 \arctan(\sqrt{x}) + c. \]
Now,
\[ \int_0^1 \frac{1}{\sqrt{x(1 + x)}} \, dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x(1 + x)}} \, dx = \lim_{a \to 0^+} \left[ 2 \arctan(\sqrt{x}) \right]_a^1 \]
\[ = \lim_{a \to 0^+} 2[\arctan(1) - \arctan(a)] = 2[\arctan(1) - \arctan(0)] \]
\[ = 2 \left( \frac{\pi}{4} - 0 \right) = \frac{\pi}{2}, \]
and
\[ \int_1^\infty \frac{1}{\sqrt{x(1 + x)}} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{x(1 + x)}} \, dx = \lim_{b \to \infty} \left[ 2 \arctan(\sqrt{x}) \right]_1^b \]
\[ = \lim_{b \to \infty} 2 \left[ \arctan(\sqrt{b}) - \arctan(1) \right] = 2 \left( \frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi}{2}. \]
Since
\[ \int_0^1 \frac{1}{\sqrt{x(1 + x)}} \, dx \quad \text{and} \quad \int_1^\infty \frac{1}{\sqrt{x(1 + x)}} \, dx \]
both converge, we conclude that
\[ \int_0^\infty \frac{1}{\sqrt{x(1 + x)}} \, dx = \int_0^1 \frac{1}{\sqrt{x(1 + x)}} \, dx + \int_1^\infty \frac{1}{\sqrt{x(1 + x)}} \, dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi \]
by Definition [7.11].

We conclude this section with one more result concerning improper integrals called the Comparison Test for Integrals, which will be needed in §7.2.

**Theorem 7.13 (Comparison Test for Integrals).** Suppose \( f \in R[a, x] \) for all \( x \geq a \), and \( 0 \leq f \leq g \) on \([a, \infty)\). If \( \int_a^\infty g \) is convergent, then \( \int_a^\infty f \) is convergent.
The Laplace transform is a function \( \mathcal{L} \) whose domain consists of functions of a particular kind, and whose range also consists of functions. If \( f \in \text{Dom}(\mathcal{L}) \) and \( F \) is the function that \( \mathcal{L} \) returns as “output” when given \( f \) as “input,” we write \( \mathcal{L}[f] = F \). If \( f \) is a function of \( t \) and \( F \) is a function of \( s \), then we may also write \( \mathcal{L}[f(t)] = F(s) \), which is taken to mean the same thing since all we are doing is replacing the name of a function by its defining rule. Since \( \mathcal{L}[f] \) is a function of \( s \) it makes sense to write \( \mathcal{L}[f](s) \), which is usually done in situations when a symbol for the Laplace transform of \( f \) (such as \( F \)) has not been given.

**Definition 7.14.** Given a function \( f : [0, \infty) \to \mathbb{R} \), the Laplace transform of \( f \) is the function \( \mathcal{L}[f] \) given by

\[
\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t) \, dt \tag{3}
\]

for all \( s \in \mathbb{R} \) for which the integral converges.

As the definition implies, the domain of \( \mathcal{L}[f] \) is the set of all \( s \in \mathbb{R} \) such that the improper integral in (3) exists as a real number. Recall that

\[
\int_0^\infty e^{-st} f(t) \, dt = \lim_{b \to \infty} \int_0^b e^{-st} f(t) \, dt,
\]

so any question concerning whether the improper integral exists boils down to a question about whether the associated limit of Riemann integrals exists.

**Example 7.15.** Find \( \mathcal{L}[t^2] \), the Laplace transform of the function \( t \mapsto t^2 \).

**Solution.** To “find” \( \mathcal{L}[t^2] \) means to find an expression for \( \mathcal{L}[t^2](s) \). By definition we have

\[
\mathcal{L}[t^2](s) = \int_0^\infty e^{-st} t^2 \, dt = \lim_{b \to \infty} \int_0^b t^2 e^{-st} \, dt. \tag{4}
\]

To evaluate the definite integral we employ integration by parts. Letting \( u(t) = t^2 \) and \( v'(t) = e^{-st} \), we obtain \( u'(t) = 2t \) and \( v(t) = -e^{-st}/s \), and so

\[
\int_0^b t^2 e^{-st} \, dt = \left[ -\frac{1}{s} t^2 e^{-st} \right]_0^b - \int_0^b \frac{2}{s} t e^{-st} \, dt = -\frac{1}{s} b^2 e^{-sb} + \frac{2}{s} \int_0^b t e^{-st} \, dt. \tag{5}
\]

Now, the integral \( \int_0^b t e^{-st} \, dt \) itself requires integration by parts. Letting \( u(t) = t \) and \( v'(t) = e^{-st} \), we obtain \( u'(t) = 1 \) and \( v(t) = -e^{-st}/s \), and so

\[
\int_0^b t e^{-st} \, dt = \left[ -\frac{1}{s} te^{-st} \right]_0^b - \int_0^b \frac{1}{s} e^{-st} \, dt = -\frac{1}{s} be^{-sb} - \frac{1}{s^2} \left[ e^{-st} \right]_0^b
\]

\[
= -\frac{1}{s} be^{-sb} - \frac{1}{s^2} (e^{-sb} - 1). \tag{6}
\]
Putting (6) into (5) gives
\[
\int_0^b t^2 e^{-st} \, dt = -\frac{1}{s} b^2 e^{-sb} + 2 \left[ -\frac{1}{s} b e^{-sb} - \frac{1}{s^2} (e^{-sb} - 1) \right] = \frac{2}{s^3} - \frac{s^2 b^2 + 2sb + 2}{s^3 e^{sb}},
\]
and putting this result into (4) yields
\[
\mathcal{L}[t^2](s) = \lim_{b \to \infty} \left( \frac{2}{s^3} - \frac{s^2 b^2 + 2sb + 2}{s^3 e^{sb}} \right).
\]
This limit does not exist if \( s \leq 0 \); however if \( s > 0 \), then by two successive applications of L'Hôpital’s Rule we obtain
\[
\lim_{b \to \infty} \frac{s^2 b^2 + 2sb + 2}{s^3 e^{sb}} = \lim_{b \to \infty} \frac{2s^2}{s^4 e^{sb}} = \lim_{b \to \infty} \frac{2}{s^3} = 0,
\]
and therefore
\[
\mathcal{L}[t^2](s) = \lim_{b \to \infty} \frac{2}{s^3} - \lim_{b \to \infty} \frac{s^2 b^2 + 2sb + 2}{s^3 e^{sb}} = \frac{2}{s^3}
\]
for all \( s > 0 \).

One key feature of the Laplace transform is that when it is applied to a piecewise-defined function of \( t \), the result is a function of \( s \) that is defined by a single expression.

**Example 7.16.** Find the Laplace transform of the function
\[
f(t) = \begin{cases} 
1 - t, & \text{if } 0 \leq t \leq 1 \\
0, & \text{if } 1 < t \leq 3 \\
e^{2t}, & \text{if } 3 < t < \infty
\end{cases}
\]

**Solution.** Using the usual properties of the Riemann integral established in calculus and assuming \( s \neq 2 \), we obtain
\[
\mathcal{L}[f](s) = \lim_{T \to \infty} \int_0^T e^{-st} f(t) \, dt \\
= \lim_{T \to \infty} \left( \int_0^1 e^{-st}(1 - t) \, dt + \int_1^3 e^{-st} \cdot 0 \, dt + \int_3^T e^{-st} e^{2t} \, dt \right) \\
= \lim_{T \to \infty} \left( \int_0^1 e^{-st} \, dt - \int_0^1 te^{-st} \, dt + \int_3^T e^{(2-s)t} \, dt \right) \\
= \lim_{T \to \infty} \left[ -\frac{1}{s} (e^{-s} - 1) - \left( -\frac{1}{s} e^{-s} - \frac{1}{s^2} (e^{-s} - 1) \right) + \frac{1}{2 - s} \left( e^{(2-s)T} - e^{3(2-s)} \right) \right] \\
= \lim_{T \to \infty} \left[ \frac{e^{-s} + s - 1}{s^2} + \frac{1}{2 - s} (e^{(2-s)T} - e^{3(2-s)}) \right]
\]

(7)
The limit does not exist if \( s < 2 \), since we would find that \( e^{(2-s)T} \to \infty \) as \( T \to \infty \). If \( s = 2 \) the limit again does not exist, since in this case
\[
\int_3^T e^{-st} e^{2t} \, dt = \int_3^T e^{-2t} e^{2t} \, dt = \int_3^T dt = T - 3,
\]
where $T - 3 \to \infty$ as $T \to \infty$. However if $s > 2$, then $e^{(2-s)T} \to 0$ as $T \to \infty$, and from (7) we obtain
\[
\mathcal{L}[f](s) = \frac{e^{-s} + s - 1}{s^2} + \frac{1}{2 - s} \left[ 0 - e^{3(2-s)} \right] = \frac{e^{-s} + s - 1}{s^2} + \frac{e^{6-3s}}{s-2},
\]
which can be seen to be not piecewise-defined! ■

The next proposition establishes that the Laplace transform is a linear transformation, also known as a linear operator.

**Proposition 7.17.** Let $f, g : [0, \infty) \to \mathbb{R}$ be functions such that $\mathcal{L}[f](s)$ and $\mathcal{L}[g](s)$ are defined for all $s > \alpha$, and let $c \in \mathbb{R}$. Then on $(\alpha, \infty)$

1. $\mathcal{L}[f + g] = \mathcal{L}[f] + \mathcal{L}[g]$
2. $\mathcal{L}[cf] = c\mathcal{L}[f]$

**Proof.**

**Proof of Part (1).** Let $s > \alpha$. Then $\mathcal{L}[f](s)$ and $\mathcal{L}[g](s)$ are both defined, which is to say that the limits
\[
\lim_{T \to \infty} \int_0^T e^{-st} f(t) \, dt \quad \text{and} \quad \lim_{T \to \infty} \int_0^T e^{-st} g(t) \, dt
\]
both exist. This fact allows us to use a limit law to write
\[
\mathcal{L}[f](s) + \mathcal{L}[g](s) = \lim_{T \to \infty} \int_0^T e^{-st} f(t) \, dt + \lim_{T \to \infty} \int_0^T e^{-st} g(t) \, dt
\]
\[
= \lim_{T \to \infty} \left[ \int_0^T e^{-st} f(t) \, dt + \int_0^T e^{-st} g(t) \, dt \right],
\]
and thus by an established property of the Riemann integral we obtain
\[
\mathcal{L}[f](s) + \mathcal{L}[g](s) = \lim_{T \to \infty} \left[ \int_0^T (e^{-st} f(t) \, dt + e^{-st} g(t)) \, dt \right]
\]
\[
= \int_0^T e^{-st} (f + g)(t) \, dt = \mathcal{L}[f + g](s),
\]
where of course $(f + g)(t) = f(t) + g(t)$ by definition. Hence
\[
(\mathcal{L}[f] + \mathcal{L}[g])(s) = \mathcal{L}[f](s) + \mathcal{L}[g](s) = \mathcal{L}[f + g](s)
\]
for all $s > \alpha$, which proves part (1).

**Proof of Part (2).** Done similarly, and so left as an exercise. ■

Table 7[1] gives the Laplace transforms for many frequently encountered functions. (A more comprehensive table can be found at the end of the chapter.) Indeed the table, together with Proposition 7.17, allows for easy computation of the transforms of most functions that arise in applications. To find the transform of $\sin bt$ or $\cos bt$, let $a = 0$ in $e^{at} \sin bt$ or $e^{at} \cos bt$, respectively. For the transform of $t^2$, let $a = 0$ and $n = 2$ in $e^{atn}$ to obtain
\[
\mathcal{L}[t^2](s) = \frac{2!}{(s - 0)^{2+1}} = \frac{2}{s^3}
\]
as found in Example 7.15. For the transform of 1, let \( a = 0 \) and \( n = 0 \) in \( e^{at}t^n \) to obtain
\[
L[1](s) = \frac{0!}{(s - 0)^{0+1}} = \frac{1}{s}
\]
from the table.

**Example 7.18.** Find \( L[f] \) for \( f(t) = e^{-5t} \sin \pi t - \cosh 8t + 7 \).

**Solution.** Recall that the hyperbolic cosine function \( \cosh(x) \) is defined by \( \cosh(x) = \frac{1}{2}(e^x + e^{-x}) \). By Proposition 7.17 we obtain
\[
L[f](s) = L[e^{-5t} \sin \pi t - \cosh 8t + 7](s) \\
= L[e^{-5t} \sin \pi t](s) - L[\cosh 8t](s) + L[7](s) \\
= L[e^{-5t} \sin \pi t](s) - \frac{1}{2}L[e^8t](s) + \frac{1}{2}L[e^{-8t}](s) + 7L[1](s) \\
= \frac{\pi}{(s + 5)^2 + \pi^2}
\]
for $s > \pi$, 
$$
\mathcal{L}[e^{st}](s) = \frac{1}{s - 8}
$$
for $s > 8$, 
$$
\mathcal{L}[e^{-st}](s) = \frac{1}{s + 8}
$$
for $s > -8$, and finally $\mathcal{L}[1](s) = 1/s$ for $s > 0$. Putting all these results into (8) gives 
$$
\mathcal{L}[f](s) = \pi \left(\frac{1}{s + 5}\right)^2 + \pi^2 - \frac{1}{2} \cdot \frac{1}{s - 8} - \frac{1}{2} \cdot \frac{1}{s + 8} + 7 \cdot \frac{1}{s}
$$
for $s > 8$, which is the intersection of the domains of the individual transforms. 

Not all functions have a Laplace transform. The remainder of this section will be devoted to developing some results that will help determine whether or not a given function has a transform.

**Definition 7.19.** Let $-\infty < a < b < \infty$. A function $f$ is **piecewise continuous on** $[a, b]$ if it satisfies the following:
1. $f$ is continuous on $[a, b]$ except at a finite number of points, at most.
2. At each $c \in (a, b)$ where $f$ is discontinuous the limits $\lim_{t \to c^+} f(t)$ and $\lim_{t \to c^-} f(t)$ both exist in $\mathbb{R}$.
3. If $f$ is discontinuous at $a$, then $\lim_{t \to a^+} f(t) \in \mathbb{R}$; and if $f$ is discontinuous at $b$, then $\lim_{t \to b^-} f(t) \in \mathbb{R}$.

A function $f$ is **piecewise continuous on** $[a, \infty)$ if it is piecewise continuous on $[a, c]$ for all $c > a$.

A careful reading of Definition 7.19 should make it clear that in order for $f$ to be piecewise continuous on $[a, b]$ or $[a, \infty)$, it is not required that $f(t)$ be defined for all $t$ in $[a, b]$ or $[a, \infty)$, respectively. This is because a function is considered to be discontinuous at any point outside its domain. Thus, for example,

$$
f(t) = \begin{cases} 
3t^2, & \text{if } t < 5 \\
4 - 8t, & \text{if } t > 5
\end{cases}
$$

is piecewise continuous on $[0, 10]$ and $[0, \infty)$, with $f$ considered discontinuous at 5 since the domain of $f$ is $(-\infty, 5) \cup (5, \infty)$. As another example, $g(t) = 1/t$ is not piecewise continuous on $[-1, 1]$ since 

$$
\lim_{t \to 0^+} \frac{1}{t} = \infty \notin \mathbb{R},
$$

which runs afoul of part (2) of Definition 7.19.

It is a fact, easily verifiable, that a piecewise continuous function defined on a closed, bounded interval is necessarily a bounded function.

**Remark.** For our own purposes we will only work with piecewise continuous functions that are defined at every point on a closed interval of the form $[a, b]$ or $[a, \infty)$.
To prove the following proposition we need to recall two facts from calculus: first, if \( f \in \mathcal{R}[a,b] \) and \( g = f \) on \([a,b] \) except at a finite number of points, then \( g \in \mathcal{R}[a,b] \) with \( \int_a^b g = \int_a^b f \); and second, if \( f \in \mathcal{R}[a,c] \) and \( f \in \mathcal{R}[c,b] \) (where \( a < c < b \)), then \( f \in \mathcal{R}[a,b] \). The latter fact is easily extended: if

\[
f \in \mathcal{R}[t_0, t_1], \mathcal{R}[t_1, t_2], \ldots, \mathcal{R}[t_{n-1}, t_n],
\]

for \( t_0, t_1, \ldots, t_n \in \mathbb{R} \) such that \( t_0 < t_1 < \cdots < t_n \), then \( f \in \mathcal{R}[t_0, t_n] \).

**Proposition 7.20.** If \( f : [a, b] \to \mathbb{R} \) is piecewise continuous, then \( f \in \mathcal{R}[a, b] \).

**Proof.** Suppose that \( f : [a, b] \to \mathbb{R} \) is piecewise continuous. If \( f \) is continuous on \([a, b] \) then \( f \in \mathcal{R}[a, b] \) follows immediately since continuity on a closed, bounded interval implies integrability. Suppose that \( f \) has precisely \( n \) discontinuities on \([a, b] \) for some \( n \in \mathbb{N} \). Thus \( f \) has discontinuities at points \( t_1, \ldots, t_n \in [a, b] \), where \( a \leq t_1 < \cdots < t_n \leq b \). For convenience we will assume that \( t_1 \neq a \) and \( t_n \neq b \), and set \( t_0 = a \) and \( t_{n+1} = b \). Then there exist functions \( g_0, \ldots, g_n \) such that \( g_k : [t_k, t_{k+1}] \to \mathbb{R} \) and \( g_k \) is continuous on \((t_k, t_{k+1})\) for each \( k = 0, \ldots, n \), and

\[
f(t) = \begin{cases} 
g_0(t), & a \leq t < t_1 
g_1(t), & t_1 \leq t < t_2 
g_2(t), & t_2 \leq t < t_3 
\vdots & \vdots 
g_n(t), & t_n \leq t < b 
\end{cases}
\]

Fix \( 0 \leq k \leq n \). By Definition 7.19 there exist real numbers \( c_l \) and \( c_r \) such that

\[
\lim_{t \to t_k^-} f(t) = \lim_{t \to t_k^+} g_k(t) = c_l \quad \text{and} \quad \lim_{t \to t_{k+1}^-} f(t) = \lim_{t \to t_{k+1}^+} g_k(t) = c_r.
\]

Define \( \hat{g}_k : [t_k, t_{k+1}] \to \mathbb{R} \) by

\[
\hat{g}_k(t) = \begin{cases} 
c_l, & t = t_k 
g_k(t), & t_k < t < t_{k+1} 
c_r, & t = t_{k+1} 
\end{cases}
\]

Since \( \hat{g}_k \) is continuous on \([t_k, t_{k+1}]\) we have \( \hat{g}_k \in \mathcal{R}[t_k, t_{k+1}] \). Now, \( f = \hat{g}_k \) on \([t_k, t_{k+1}]\) except possibly at \( t_k \) or \( t_{k+1} \), and so by the first result from calculus mentioned above we conclude that \( f \in \mathcal{R}[t_k, t_{k+1}] \). Since \( 0 \leq k \leq n \) is arbitrary we now have

\[
f \in \mathcal{R}[t_0, t_1], \mathcal{R}[t_1, t_2], \ldots, \mathcal{R}[t_{n}, t_{n+1}],
\]

and so the second aforementioned calculus result implies that \( f \in \mathcal{R}[t_0, t_{n+1}] = \mathcal{R}[a, b] \).

**Definition 7.21.** A function \( f \) is of exponential order \( \alpha \) if there exist constants \( c, M > 0 \) such that

\[
|f(t)| \leq Me^{\alpha t}
\]

for all \( t \geq c \).
Observe from the definition that if \( f \) is of exponential order \( \alpha \), then it is in fact of exponential order \( \beta \) for any \( \beta > \alpha \), since \( Me^{\alpha t} < Me^{\beta t} \).

**Proposition 7.22.** If \( f \) is of exponential order \( \alpha \), then
\[
\lim_{t \to \infty} f(t)e^{-st} = 0
\]
for all \( s > \alpha \).

**Proof.** Suppose \( f \) is of exponential order \( \alpha \), so there are constants \( c, M > 0 \) such that \( |f(t)| \leq Me^{\alpha t} \) for all \( t \geq c \). Let \( s > \alpha \). Since
\[
0 \leq |f(t)|e^{-st} \leq Me^{\alpha t}e^{-st} = Me^{(\alpha-s)t}
\]
for all \( t \geq c \), and \( Me^{(\alpha-s)t} \to 0 \) as \( t \to \infty \), the Squeeze Theorem implies that \( |f(t)|e^{-st} \to 0 \) as \( t \to 0 \). Therefore \( f(t)e^{-st} \to 0 \) as \( t \to \infty \) as well. \( \blacksquare \)

To prove the next theorem we need yet three more facts from calculus: first, if \( f, g \in \mathcal{R}[a,b] \), then \( fg \in \mathcal{R}[a,b] \) also; second, if \( f \in \mathcal{R}[a,b] \), then \( |f| \in \mathcal{R}[a,b] \) also; and third, if \( \int_a^\infty |f| \) is convergent, then \( \int_a^\infty f \) is convergent. In addition we will need the Comparison Test for Integrals stated at the end of §7.1.

**Theorem 7.23.** If \( f \) is piecewise continuous on \([0, \infty)\) and of exponential order \( \alpha \), then \( \mathcal{L}[f](s) \) exists for all \( s > \alpha \).

**Proof.** Suppose \( f \) is piecewise continuous on \([0, \infty)\), and also of exponential order \( \alpha \) so there exists \( c, M > 0 \) such that
\[
|f(t)| \leq Me^{\alpha t}
\]
for all \( t \geq c \). An immediate consequence of Definition 7.19 is that \( f \) is piecewise continuous on \([c, t] \) for all \( t \geq c \), so that \( f \in \mathcal{R}[c, t] \) by Proposition 7.20 and hence \( |f| \in \mathcal{R}[c, t] \). We also have \( e^{-st} \in \mathcal{R}[c, t] \) by virtue of continuity, and thus the product \( e^{-st}|f(t)| \) is in \( \mathcal{R}[c, t] \) for all \( t \geq c \).

For \( t \geq c \) we have
\[
0 \leq e^{-st}|f(t)| \leq Me^{-st}e^{\alpha t} = Me^{(\alpha-s)t}.
\]

Let \( s > \alpha \). Then \( \alpha - s < 0 \) so that \( e^{(\alpha-s)t} \to 0 \) as \( t \to \infty \). Now,
\[
\int_c^\infty Me^{(\alpha-s)t}dt = \lim_{T \to \infty} \int_c^T Me^{(\alpha-s)t}dt = \lim_{T \to \infty} \frac{M}{\alpha-s} [e^{(\alpha-s)T} - e^{(\alpha-s)c}]
\]
\[
= \frac{M}{\alpha-s} [0 - e^{(\alpha-s)c}] = \frac{Me^{(\alpha-s)c}}{s-\alpha},
\]
so
\[
\int_c^\infty Me^{(\alpha-s)t}dt
\]
is convergent. Therefore
\[
\int_c^\infty e^{-st}|f(t)| dt
\]
is convergent by the Comparison Test for Integrals, whence it follows that
\[\int_{c}^{\infty} e^{-st} f(t) \, dt\]
is also convergent and hence exists in \(\mathbb{R}\).

Finally, \(f\) is piecewise continuous on \([a, c]\), so that \(f\) and subsequently \(e^{-st} f(t)\) is integrable on \([a, c]\). That is,
\[\int_{a}^{c} e^{-st} f(t) \, dt\]
is defined in \(\mathbb{R}\), and then
\[
\mathcal{L}[f](s) = \int_{a}^{\infty} e^{-st} f(t) \, dt = \lim_{T \to \infty} \int_{a}^{T} e^{-st} f(t) \, dt \\
= \lim_{T \to \infty} \left[ \int_{a}^{c} e^{-st} f(t) \, dt + \int_{c}^{T} e^{-st} f(t) \, dt \right] \\
= \int_{a}^{c} e^{-st} f(t) \, dt + \int_{c}^{\infty} e^{-st} f(t) \, dt
\]
shows that \(\mathcal{L}[f](s)\) likewise is defined in \(\mathbb{R}\).

Therefore \(\mathcal{L}[f](s)\) exists for all \(s > \alpha\), and the proof is done. ■
In this section we establish several properties of the Laplace transform which will prove to be essential labor-saving devices starting in Section 7.5.

**Theorem 7.24.** If \( \text{Dom}(L[f]) = (\alpha, \infty) \),

\[
L[e^{at}f(t)](s) = L[f(t)](s-a)
\]

for all \( s > \alpha + a \).

**Example 7.25.** From Example 7.18 we found that, for \( f(t) = e^{-5t} \sin \pi t - \cosh 8t + 7 \),

\[
L[f](s) := F(s) = \frac{\pi}{(s+5)^2 + \pi^2} - \frac{s}{s^2 - 64} + \frac{7}{s}
\]

for \( s > 8 \). Therefore, by Theorem 7.24,

\[
L[e^{7t}f(t)](s) = L[e^{2t} \sin \pi t - e^{7t} \cosh 8t + 7e^{7t}](s) = F(s-7)
\]

\[
= \frac{\pi}{((s-7) + 5)^2 + \pi^2} - \frac{s-7}{(s-7)^2 - 64} + \frac{7}{s-7}
\]

\[
= \frac{\pi}{(s-2)^2 + \pi^2} - \frac{s-7}{s^2 - 14s - 15} + \frac{7}{s-7}
\]

for all \( s > 8 + 7 = 15 \). \( \blacksquare \)

**Theorem 7.26.** For \( n \geq 1 \), suppose \( f, f', \ldots, f^{(n)} \) are of exponential order \( \alpha \). If \( f, f', \ldots, f^{(n-1)} \) are continuous on \([0, \infty)\), and \( f^{(n)} \) is piecewise continuous on \([0, \infty)\), then

\[
L[f^{(n)}](s) = s^n L[f](s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)
\]

for all \( s > \alpha \).

**Proof.** When \( n = 1 \) the theorem states that if \( f \) and \( f' \) are of exponential order \( \alpha \), \( f \) is continuous on \([0, \infty)\), and \( f' \) is piecewise continuous on \([0, \infty)\), then

\[
L[f'](s) = sL[f](s) - f(0)
\]

for all \( s > \alpha \). We prove this to start. Fix \( s > \alpha \). Integration by parts gives

\[
L[f'](s) = \lim_{T \to \infty} \int_{0}^{T} f'(t)e^{-st} \, dt = \lim_{T \to \infty} \left( [f(t)e^{-st}]_{0}^{T} - \int_{0}^{T} -sf(t)e^{-st} \, dt \right)
\]

\[
= \lim_{T \to \infty} \left( f(T)e^{-sT} - f(0) + s \int_{0}^{T} f(t)e^{-st} \, dt \right) = -f(0) + sL[f](s),
\]

where \( f(T)e^{-sT} \to 0 \) as \( T \to \infty \) by Proposition 7.22. This shows that the statement of the theorem holds in the case when \( n = 1 \).
When \( n = 2 \) the theorem states that if \( f, f', \) and \( f'' \) are of exponential order \( \alpha \), \( f \) and \( f' \) are continuous on \([0, \infty)\), and \( f'' \) is piecewise continuous on \([0, \infty)\), then
\[
\mathcal{L}[f''](s) = s^2 \mathcal{L}[f](s) - sf(0) - f'(0)
\] (10)
for all \( s > \alpha \). To show this, we simply apply the \( n = 1 \) case first to \( f' \), and then to \( f \):
\[
\mathcal{L}[f''](s) = \mathcal{L}[(f')'](s) = s \mathcal{L}[f'](s) - f'(0) = s \left[s \mathcal{L}[f](s) - f(0) \right] - f'(0)
\]
\[
= s^2 \mathcal{L}[f](s) - sf(0) - f'(0).
\]

The full proof of the theorem is done by induction: suppose the theorem is true for some arbitrary \( n \geq 1 \), and then show that it is true for \( n + 1 \).

**Theorem 7.27.** Suppose that \( f \) is piecewise continuous on \([0, \infty)\) and of exponential order \( \alpha \). If \( \mathcal{L}[f] = F \), then
\[
\mathcal{L}[t^n f(t)](s) = (-1)^n F^{(n)}(s)
\]
for all \( s > \alpha \).
7.4 – The Inverse Laplace Transform

Definition 7.28. Given a function \( F : [\alpha, \infty) \rightarrow \mathbb{R} \), if there exists a continuous function \( f : [0, \infty) \rightarrow \mathbb{R} \) such that \( \mathcal{L}[f](s) = F(s) \) for all \( s > \alpha \), then we call \( f \) the inverse Laplace transform of \( F \) and write \( f = \mathcal{L}^{-1}[F] \).

Table 7 can be used to find the inverse Laplace transform of any rational function \( p(s)/q(s) \) for which \( \deg(p) < \deg(q) \) and \( q \) can be written as a product of linear or quadratic factors with real coefficients. This will often necessitate application of the partial fraction decomposition procedure, however.

Theorem 7.29 (Partial Fraction Decomposition). Let \( p \) and \( q \) be polynomial functions such that \( \deg(p) < \deg(q) \), and suppose \( q \) can be factored as a product of polynomials of degree at most 2. Then one of the following cases must hold.

1. \( q(s) \) has form

\[
q_1(s) = (a_1s + b_1)(a_2s + b_2) \cdots (a_ns + b_n)
\]

\( a_is + b_i \neq a_js + b_j \) whenever \( i \neq j \). So \( q_1(s) \) is a product of distinct linear factors. Then we find constants \( A_1, A_2, \ldots, A_n \) for which

\[
p(s) = A_1(a_1s + b_1) + A_2(a_2s + b_2) + \cdots + A_n(a_ns + b_n). \tag{11}
\]

2. \( q(s) \) has form

\[
q_2(s) = (as + b)^n
\]

for some integer \( n \geq 2 \). So \( q_2(s) \) is a product of repeated linear factors. Then we find constants \( B_1, B_2, \ldots, B_n \) for which

\[
p(s) = B_1(as + b) + B_2(as + b)^2 + \cdots + B_n(as + b)^n. \tag{12}
\]

3. \( q(s) \) has form

\[
q_3(s) = (a_1s^2 + b_1s + c_1) \cdots (a_ns^2 + b_ns + c_n),
\]

with \( b_i^2 - 4a_ic_i < 0 \) for each \( i \), and \( a_is^2 + b_is + c_i \neq a_js^2 + b_js + c_j \) if \( i \neq j \). So \( q_3(s) \) is a product of distinct irreducible quadratic factors. We find constants \( C_1, \ldots, C_n \) and \( D_1, \ldots, D_n \) for which

\[
p(s) = C_1s + D_1 + C_2s + D_2 + \cdots + C_n s + D_n. \tag{13}
\]

4. \( q(s) \) has form

\[
q_4(s) = (as^2 + bs + c)^n
\]

with \( b^2 - 4ac < 0 \) and \( n \geq 2 \). So \( q_4(s) \) is a product of repeated irreducible quadratic factors. We find constants \( C_1, \ldots, C_n \) and \( D_1, \ldots, D_n \) for which

\[
p(s) = C_1s + D_1 + C_2s + D_2 + \cdots + C_n s + D_n. \tag{14}
\]
5. \( q(s) \) has form

\[ q_5(s) = q_1(s)q_2(s)q_3(s)q_4(s). \]

Then the decomposition is given by

\[
\frac{p(s)}{q_5(s)} = \frac{p_1(s)}{q_1(s)} + \frac{p_2(s)}{q_2(s)} + \frac{p_3(s)}{q_3(s)} + \frac{p_4(s)}{q_4(s)},
\]

(15)

where \( \frac{p_1(s)}{q_1(s)}, \frac{p_2(s)}{q_2(s)}, \frac{p_3(s)}{q_3(s)}, \) and \( \frac{p_4(s)}{q_4(s)} \) are given by the right-hand sides of equations (11), (12), (13), and (14), respectively.

**Example 7.30.** Find \( \mathcal{L}^{-1}[F] \), where

\[
F(s) = \frac{6s^2 + 5s - 3}{s^3 + 2s^2 - 3s}
\]

**Solution.** Factoring the denominator yields \( s(s + 3)(s - 1) \), which are three distinct linear factors and so Case (1) of Theorem 7.29 applies here:

\[
F(s) = \frac{6s^2 + 5s - 3}{s(s + 3)(s - 1)} = \frac{A_1}{s} + \frac{A_2}{s + 3} + \frac{A_3}{s - 1}.
\]

Multiplying both sides by \( s(s + 3)(s - 1) \), we obtain

\[
6s^2 + 5s - 3 = A_1(s + 3)(s - 1) + A_2s(s - 1) + A_3s(s + 3)
\]

\[
= (A_1s^2 + 2A_1s - 3A_1) + (A_2s^2 - A_2s) + (A_3s^2 + 3A_3s)
\]

\[
= (A_1 + A_2 + A_3)s^2 + (2A_1 - A_2 + 3A_3)s - 3A_1
\]

Equating coefficients of \( s^2 \), coefficients of \( s \), and constant terms, we obtain a system of equations,

\[
\begin{aligned}
A_1 + A_2 + A_3 &= 6 \\
2A_1 - A_2 + 3A_3 &= 5 \\
3A_1 &= 3
\end{aligned}
\]

From the third equation we obtain \( A_1 = 1 \). Putting this into the first equation yields \( 1 + A_2 + A_3 = 6 \), and so \( A_2 = 5 - A_3 \). Now from the second equation we have

\[
2(1) - (5 - A_3) + 3A_3 = 5 \Rightarrow 4A_3 - 3 = 5 \Rightarrow A_3 = 2,
\]

and thus \( A_2 = 5 - A_3 = 3 \). We now have

\[
F(s) = \frac{1}{s} + \frac{3}{s + 3} + \frac{2}{s - 1},
\]

and so

\[
\mathcal{L}^{-1}[F](t) = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{3}{s + 3} + \frac{2}{s - 1}\right](t)
\]

\[
= \mathcal{L}^{-1}\left[\frac{1}{s}\right](t) + \mathcal{L}^{-1}\left[\frac{3}{s + 3}\right](t) + \mathcal{L}^{-1}\left[\frac{2}{s - 1}\right](t)
\]

\[
= 1 + 3e^{-3t} + 2e^t,
\]
Example 7.31. Find $L^{-1}[F]$, where

$$F(s) = \frac{s^2}{(s + 1)^3}$$

Solution. Here we have a repeated linear factor, and so in accordance with Case (2) of Theorem 7.29 we obtain

$$\frac{s^2}{(s + 1)^3} = \frac{B_1}{s + 1} + \frac{B_2}{(s + 1)^2} + \frac{B_3}{(s + 1)^3}.$$ 

Multiplying both sides by $(s + 1)^3$ yields

$$s^2 = B_1(s + 1)^2 + B_2(s + 1) + B_3,$$

whence we obtain

$$s^2 = B_1 s^2 + (2B_1 + B_2) s + (B_1 + B_2 + B_3).$$

Equating coefficients of matching powers of $s$ produces the system of equations

$$\begin{cases}
B_1 = 1 \\
2B_1 + B_2 = 0 \\
B_1 + B_2 + B_3 = 0
\end{cases}$$

Putting $B_1 = 1$ from the first equation into the second equation gives $2 + B_2 = 0$, or $B_2 = -2$. Now the third equation becomes $1 - 2 + B_3 = 0$, or $B_3 = 1$. Hence

$$L^{-1}[F](t) = L^{-1} \left[ \frac{1}{s + 1} - \frac{2}{(s + 1)^2} + \frac{1}{(s + 1)^3} \right](t)$$

$$= L^{-1} \left[ \frac{0!}{s + 1} - 2 \cdot \frac{1!}{(s + 1)^2} + \frac{1}{2} \cdot \frac{2!}{(s + 1)^3} \right](t)$$

$$= L^{-1} \left[ \frac{0!}{s + 1} \right](t) - 2L^{-1} \left[ \frac{1!}{(s + 1)^2} \right](t) + \frac{1}{2} L^{-1} \left[ \frac{2!}{(s + 1)^3} \right](t)$$

$$= e^{-t} - 2te^{-t} + \frac{1}{2} t^2 e^{-t},$$

using the linearity properties of $L^{-1}$.

Example 7.32. Find $L^{-1}[F]$, where

$$F(s) = \frac{5s^2 + 3s - 2}{s^4 + s^3 - 2s^2}$$

Solution. Factoring the denominator gives

$$F(s) = \frac{5s^2 + 3s - 2}{s^2(s + 2)(s - 1)},$$

so $s + 2$ and $s - 1$ are distinct linear factors, and $s$ is a repeated factor. According to (5) of Theorem 7.29 we have

$$\frac{5s^2 + 3s - 2}{s^2(s + 2)(s - 1)} = \frac{P_1(s)}{(s + 2)(s - 1)} + \frac{P_2(s)}{s^2} = \left( \frac{A_1}{s + 2} + \frac{A_2}{s - 1} \right) + \left( \frac{B_1}{s} + \frac{B_2}{s^2} \right),$$
employing the prescribed decompositions for Cases (1) and (2) of Theorem 7.29. Multiplying the left and right sides of the equation by $s^2(s + 2)(s - 1)$ yields

$$5s^2 + 3s - 2 = A_1 s^2(s - 1) + A_2 s^2(s + 2) + B_1 s(s + 2)(s - 1) + B_2(s + 2)(s - 1),$$

and thus

$$5s^2 + 3s - 2 = (A_1 + A_2 + B_1)s^3 + (-A_1 + 2A_2 + B_1 + B_2)s^2 + (-2B_1 + B_2)s - 2B_2.$$ 

Equating coefficients of matching powers of $s$ produces the system of equations

$$\begin{cases}
A_1 + A_2 + B_1 = 0 \\
-A_1 + 2A_2 + B_1 + B_2 = 5 \\
-2B_1 + B_2 = 3 \\
2B_2 = 2
\end{cases}$$

The solution to the system is $A_1 = -1$, $A_2 = 2$, $B_1 = -1$, $B_2 = 1$. Hence,

$$\mathcal{L}^{-1}[F](t) = \mathcal{L}^{-1}\left[-\frac{1}{s+2} + \frac{2}{s-1} - \frac{1}{s} + \frac{1}{s^2}\right](t)$$

$$= -\mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t) + 2\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) - \mathcal{L}^{-1}\left[\frac{1}{s}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t)$$

$$= -e^{-2t} + 2e^t - 1 + t,$$

using the linearity properties of $\mathcal{L}^{-1}$.
7.5 – The Method of Laplace Transforms

Recall from Theorem 4.3 that an initial value problem of the form
\[ a_2y'' + a_1y' + a_0y = f(t), \quad y(t_0) = b_0, \quad y'(t_0) = b_1 \]
has a unique solution that is valid on \((-\infty, \infty)\). Now, if a function \( \varphi : [0, \infty) \to \mathbb{R} \) is found that satisfies the IVP on \([0, \infty)\), and \( \psi : \mathbb{R} \to \mathbb{R} \) is the unique function that satisfies the IVP on \((-\infty, \infty)\), then it must be that \( \psi(t) = \varphi(t) \) for all \( t \geq 0 \). Thus, if \( \varphi \) is known, it should be an easy matter to determine \( \psi \). To find \( \varphi \) we can use the Method of Laplace Transforms. Since
\[ a_2\varphi''(t) + a_1\varphi'(t) + a_0\varphi(t) = f(t) \]
for all \( t \in [0, \infty) \), it follows that
\[ \mathcal{L}[a_2\varphi''(t) + a_1\varphi'(t) + a_0\varphi(t)](s) = \mathcal{L}[f(t)](s) \]
for all \( s \) on some \( s \)-domain \((\alpha, \infty)\). The linearity properties of \( \mathcal{L} \) given by Proposition 7.17 then yield
\[ a_2\mathcal{L}[\varphi''(t)](s) + a_1\mathcal{L}[\varphi'(t)](s) + a_0\mathcal{L}[\varphi(t)](s) = \mathcal{L}[f(t)](s). \]

Assuming \( t_0 = 0 \) so that the initial conditions are \( \varphi(0) = b_0 \) and \( \varphi'(0) = b_1 \), and letting \( \Phi(s) = \mathcal{L}[\varphi(t)](s) \) and \( F(s) = \mathcal{L}[f(t)](s) \), by equations (9) and (10) we obtain
\[ a_2[s^2\Phi(s) - s\varphi(0) - \varphi'(0)] + a_1[s\Phi(s) - \varphi(0)] + a_0\Phi(s) = F(s), \]
and thus
\[ \Phi(s) = \frac{F(s) + (a_2s + a_1)b_0 + a_2b_1}{a_2s^2 + a_1s + a_0}. \]

Since \( \varphi(t) = \mathcal{L}^{-1}[\Phi(s)](t) \), the solution to the IVP is found as
\[ \varphi(t) = \mathcal{L}^{-1}\left[ \frac{F(s) + (a_2s + a_1)b_0 + a_2b_1}{a_2s^2 + a_1s + a_0} \right](t). \]

Examples should help illuminate the general procedure.

Example 7.33. Solve the initial value problem
\[ y'' - 4y' + 5y = 4e^{3t}, \quad y(0) = 2, \quad y'(0) = 7 \]

Solution. Taking the Laplace transform of both sides of the ODE gives
\[ \mathcal{L}[y''] - 4\mathcal{L}[y'] + 5\mathcal{L}[y] = \mathcal{L}[4e^{3t}]. \]  \hspace{1cm} (16)

Letting \( Y(s) = \mathcal{L}[y](s) \), we use equations (9) and (10), and Table 7.1 to obtain
\[ \mathcal{L}[y'](s) = s\mathcal{L}[y](s) - y(0) = sY - 2, \]
\[ \mathcal{L}[y''](s) = s^2\mathcal{L}[y](s) - sy(0) - y'(0) = s^2Y - 2s - 7, \]
and
\[ \mathcal{L}[4e^{3t}](s) = \frac{4}{s - 3}. \]
Putting these results into (16) gives
\[(s^2 Y - 2s - 7) - 4(sY - 2) + 5Y = \frac{4}{s - 3},\]
from which we get
\[(s^2 - 4s + 5)Y = \frac{4}{s - 3} + 2s - 1,\]
and finally
\[Y(s) = \frac{2s^2 - 7s + 7}{(s - 3)(s^2 - 4s + 5)}.\]

The next step is to apply partial fraction decomposition: we must determine constants \(A\), \(B\), and \(C\) so that
\[
\frac{2s^2 - 7s + 7}{(s - 3)(s^2 - 4s + 5)} = \frac{A}{s - 3} + \frac{Bs + C}{s^2 - 4s + 5}.
\]
Multiplying both sides by \((s - 3)(s^2 - 4s + 5)\) yields
\[A(s^2 - 4s + 5) + (Bs + C)(s - 3) = 2s^2 - 7s + 7,
\]
which we can rearrange to obtain
\[(A + B)s^2 + (-4A - 3B + C)s + (5A - 3C) = 2s^2 - 7s + 7.
\]
Equating coefficients, we arrive at the system of equations
\[
\begin{cases}
A + B = 2 \\
-4A - 3B + C = -7 \\
5A - 3C = 7
\end{cases}
\]
The solution to the system is \(A = 2\), \(B = 0\), and \(C = 1\). Thus,
\[\mathcal{L}[y](s) = Y(s) = \frac{2s^2 - 7s + 7}{(s - 3)(s^2 - 4s + 5)} = \frac{2}{s - 3} + \frac{1}{s^2 - 4s + 5},\]
and so
\[y(t) = \mathcal{L}^{-1}\left[\frac{2}{s - 3} + \frac{1}{s^2 - 4s + 5}\right](t) = 2\mathcal{L}^{-1}\left[\frac{1}{s - 3}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s^2 - 4s + 5}\right](t).
\]
Using Table 7\[1\] then, we at last obtain
\[y(t) = 2e^{3t} + e^{2t} \sin t\]
as the solution to the IVP.

Example 7.34. Solve the initial value problem
\[y'' + y = t, \quad y(\pi) = 0, \quad y'(\pi) = 0.\]
Solution. Equations (9) and (10) require initial conditions at \( t = 0 \), whereas here we have initial conditions at \( t = \pi \). However, if we let \( w(t) = y(t + \pi) \), then \( w'(t) = y'(t + \pi) \), \( w''(t) = y''(t + \pi) \), and also
\[
 w(0) = y(\pi) = 0 \quad \text{and} \quad w'(0) = y'(\pi) = 0.
\]
In the ODE
\[
y''(t) + y(t) = t
\]
we substitute \( t + \pi \) for \( t \) to obtain
\[
y''(t + \pi) + y(t + \pi) = t + \pi,
\]
and so arrive at the IVP
\[
w''(t) + w(t) = t + \pi, \quad w(0) = 0, \quad w'(0) = 0.
\]
We solve this IVP by the Method of Laplace Transforms as usual: letting \( W(s) = \mathcal{L}[w(t)](s) \), we have
\[
[s^2W(s) - sw(0) - w'(0)] + W(s) = \mathcal{L}[t + \pi](s) + \pi \mathcal{L}[1](s),
\]
which implies that
\[
s^2W(s) + W(s) = \frac{1}{s^2} + \frac{\pi}{s},
\]
and finally
\[
W(s) = \frac{1}{s^2(s^2 + 1)} + \frac{\pi}{s(s^2 + 1)} = \frac{1 + \pi s}{s^2(s^2 + 1)}.
\]
We must find constants \( A, B, C, \) and \( D \) such that
\[
\frac{1 + \pi s}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 1},
\]
or equivalently
\[
1 + \pi s = (A + C)s^3 + (B + D)s^2 + As + B.
\]
Clearly we must have \( A = \pi, \ B = 1, \ B + D = 0, \) and \( A + C = 0 \). The unique solution is \( (A, B, C, D) = (\pi, 1, -\pi, -1) \), and so
\[
W(s) = \frac{\pi}{s} + \frac{1}{s^2} - \frac{\pi s + 1}{s^2 + 1}.
\]
Taking the inverse Laplace transform of both sides yields
\[
w(t) = \pi \mathcal{L}^{-1}\left[\frac{1}{s}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) - \pi \mathcal{L}^{-1}\left[\frac{s}{s^2 + 1}\right](t) - \mathcal{L}^{-1}\left[\frac{1}{s^2 + 1}\right](t)
\]
\[
= \pi + t - \pi \cos(t) - \sin(t),
\]
and hence
\[
y(t + \pi) = \pi + t - \pi \cos(t) - \sin(t).
\]
Substituting \( t - \pi \) for \( t \) leads to
\[
y(t) = \pi + (t - \pi) - \pi \cos(t - \pi) - \sin(t - \pi).
\]
We simplify to obtain
\[
y(t) = t + \pi \cos(t) + \sin(t)\]
as the solution to the original IVP.

The Method of Laplace Transforms applies just as well to solving initial value problems of the form
\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(t), \quad y(0) = b_0, \ldots, y^{(n-1)}(0) = b_{n-1} \]
for \( n > 2 \) (or even \( n = 1 \)).

**Example 7.35.** Solve the initial value problem
\[ y''' - y'' + y' - y = 0, \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 3. \]

**Solution.** Taking the Laplace transform of both sides of the ODE and using Theorem 7.26 yields
\[
(s^3 \mathcal{L}[y](s) - s^2 y(0) - sy'(0) - y''(0)) - (s^2 \mathcal{L}[y](s) - sy(0) - y'(0))
+ (s \mathcal{L}[y](s) - y(0)) - \mathcal{L}[y](s) = \mathcal{L}[0](s).
\]
Letting \( Y(s) = \mathcal{L}[y(t)](s) \) and noting that \( \mathcal{L}[0](s) = 0 \), we use the initial conditions to obtain
\[
[s^3 Y(s) - s^2 - s - 3] - [s^2 Y(s) - s - 1] + [s Y(s) - 1] - Y(s) = 0,
\]
and thus
\[
Y(s) = \frac{s^2 + 3}{s^3 - s^2 + s - 1} = \frac{s^2 + 3}{(s - 1)(s^2 + 1)}.
\]
The partial fraction decomposition of the rational expression on the right-hand has the form
\[
\frac{s^2 + 3}{(s - 1)(s^2 + 1)} = \frac{A}{s - 1} + \frac{Bs + C}{s^2 + 1},
\]
whence
\[
s^2 + 3 = A(s^2 + 1) + (Bs + C)(s - 1) = (A + B)s^2 + (C - B)s + (A - C).
\]
This gives rise to the system
\[
\begin{cases}
A + B = 1 \\
-B + C = 0 \\
A - C = 3
\end{cases}
\]
which has solution \((A, B, C) = (2, -1, -1)\), and so
\[
Y(s) = \frac{2}{s - 1} - \frac{s + 1}{s^2 + 1} = \frac{2}{s - 1} - \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}.
\]
Finally,
\[
y(t) = 2\mathcal{L}^{-1}\left[\frac{1}{s - 1}\right](t) - \mathcal{L}^{-1}\left[\frac{s}{s^2 + 1}\right](t) - \mathcal{L}^{-1}\left[\frac{1}{s^2 + 1}\right](t)
\]
leads to
\[
y(t) = 2e^t - \cos t - \sin t
\]
as the solution to the IVP.
The Laplace transform can actually be employed to solve initial value problems that cannot be solved using the Method of Undetermined Coefficients. In particular there are initial value problems for which the ODE has either a piecewise-defined nonhomogeneity as illustrated in the next section, or nonconstant coefficients as illustrated in the next example.

**Example 7.36.** Solve the initial value problem

\[ ty'' - ty' + y = 2, \quad y(0) = 2, \quad y'(0) = -1 \]

**Solution.** Taking the Laplace transform of both sides of the ODE gives

\[ \mathcal{L}[ty''] - \mathcal{L}[ty'] + \mathcal{L}[y] = \mathcal{L}[2]. \]  

(17)

Letting \( Y(s) = \mathcal{L}[y](s) \), we use equations (9) and (10) to obtain

\[ \mathcal{L}[y'](s) = s\mathcal{L}[y](s) - y(0) = sY - 2 := Z_1 \]

and

\[ \mathcal{L}[y''](s) = s^2\mathcal{L}[y](s) - sy(0) - y'(0) = s^2Y - 2s + 1 := Z_2. \]

Now, by Theorem 7.27,

\[ \mathcal{L}[ty'](s) = (-1)^1Z_1' = -(sY - 2)' = -sY' - Y \]

and

\[ \mathcal{L}[ty''](s) = (-1)^1Z_2' = -(s^2Y - 2s + 1)' = -s^2Y' - 2sY + 2. \]

Putting these results into (17) yields

\[ (-s^2Y' - 2sY + 2) - (-sY' - Y) + Y = \frac{2}{s}. \]

With a little algebra the equation becomes

\[ (s - s^2)Y' + (2 - 2s)Y = \frac{2}{s} - 2, \]

which is a linear first-order differential equation. Dividing by \( s - s^2 \) puts it into standard form:

\[ Y' + \frac{2}{s}Y = \frac{2}{s^2}. \]

A suitable integrating factor is given by

\[ \mu(s) = e^{\int \frac{2}{s} ds} = s^2. \]

Multiplying the equation by \( s^2 \) yields \( s^2Y' + 2sY = 2 \), whence \( (s^2Y)' = 2 \) and so

\[ s^2Y = 2s + c \]

for arbitrary constant \( c \).

We now have

\[ \mathcal{L}[y](s) = Y(s) = \frac{2}{s} + \frac{c}{s^2}, \]

and so

\[ y(t) = \mathcal{L}^{-1}[Y](t) = 2\mathcal{L}^{-1}\left[ \frac{1}{s} \right](t) + c\mathcal{L}^{-1}\left[ \frac{1}{s^2} \right](t) = 2 + ct. \]
Thus $y'(t) = c$, and to determine $c$ we must, oddly enough, make use of the initial condition $y'(0) = -1$ again to obtain $c = -1$. Therefore

$$y(t) = 2 - t$$

is the solution to the IVP. And there was much rejoicing throughout the kingdom.
We begin with an important discontinuous function often used to model processes that change suddenly in value.

**Definition 7.37.** The **unit step function** \( u(t) \) is defined by

\[
u(t) = \begin{cases} 
0, & \text{if } t < 0 \\
1, & \text{if } t \geq 0
\end{cases}
\]

Some authors have \( > \) where we have \( \geq \), and so leave \( u(0) \) undefined; but for theoretical reasons it makes more sense to set \( u(0) = 1 \). So \( u \) has a jump discontinuity at \( t = 0 \). If we wish for a function that has a jump discontinuity at \( t = a \), we need only compose \( u \) with the function \( h(t) = t - a \):

\[(u \circ h)(t) = u(h(t)) = u(t - a) = \begin{cases} 
0, & \text{if } t < a \\
1, & \text{if } t \geq a
\end{cases}
\]

See Figure 3.

In general, given any function \( f(t) : [a, \infty) \rightarrow \mathbb{R} \), we define

\[f(t)u(t - a) = \begin{cases} 
0, & \text{if } t < a \\
f(t), & \text{if } t \geq a
\end{cases}
\]

We take this to be the case even if \( f(t) \) is undefined for \( t < a \!\).\(^2\)

**Proposition 7.38.** Let \( f : [0, \infty) \rightarrow \mathbb{R} \) be a function. If \( a \geq 0 \) and \( \mathcal{L}[f(t)](s) \) exists for \( s > \alpha \), then

\[
\mathcal{L}[f(t - a)u(t - a)](s) = e^{-as}\mathcal{L}[f(t)](s)
\]

for \( s > \alpha \).

**Proof.** Suppose that \( a \geq 0 \) and \( \mathcal{L}[f(t)](s) \) exists for all \( s > \alpha \). Thus, for any real-valued quantity \( c \) that is independent of \( t \), we have

\[
\int_0^\infty e^{-st} \cdot cf(t) \, dt = c \int_0^\infty e^{-st} f(t) \, dt
\]

\(^2\)Otherwise we’re left to interpret, say, \( \int_0^2 u(t) \, dt \) as an improper integral of the second kind, as discussed in §7.1.

\[\begin{array}{c}
u(t) \\
1 \\
\end{array} \]

\[\begin{array}{c}
t \\
\end{array} \]

**Figure 2.** The unit step function.
for any $s > \alpha$, a result we shall make use of presently.

By Definition 7.14, and making the substitution $\tau = t - a$, we obtain

$$
L[f(t-a)u(t-a)](s) = \int_0^\infty e^{-st} f(t-a)u(t-a) \, dt = \lim_{b \to \infty} \int_0^b e^{-st} f(t-a)u(t-a) \, dt
$$

$$
= \lim_{b \to \infty} \int_a^b e^{-st} f(t-a) \, dt = \lim_{b \to \infty} \int_0^{b-a} e^{-s(\tau+a)} f(\tau) \, d\tau
$$

$$
= \lim_{b \to \infty} \int_0^{b-a} e^{-s\tau} \cdot e^{-sa} f(\tau) \, d\tau = \int_0^\infty e^{-s\tau} \cdot e^{-sa} f(\tau) \, d\tau
$$

$$
= \int_0^\infty e^{-st} \cdot e^{-sa} f(t) \, dt = e^{-sa} \int_0^\infty e^{-st} f(t) \, dt
$$

$$
= e^{-as} L[f(t)](s)
$$

for any $s > \alpha$. 

In particular if we suppose that $a > 0$ and $f(t) \equiv 1$, then from (18) we obtain

$$
L[u(t-a)](s) = e^{-as} L[1](s) = \frac{e^{-as}}{s}.
$$

for all $s > 0$, which may also be obtained easily enough from Definition 7.14.

From the proposition above the following quite similar result obtains, which is needed often in applications.

**Corollary 7.39.** If $a > 0$ and $g : [a, \infty) \to \mathbb{R}$ is a function for which $L[g(t+a)](s)$ exists for $s > \alpha$, then

$$
L[g(t)u(t-a)](s) = e^{-as} L[g(t+a)](s)
$$

(19)

for $s > \alpha$.

**Proof.** Define $f : [0, \infty) \to \mathbb{R}$ by $f(t) = g(t+a)$, so that $g(t) = f(t-a)$ for $t \geq a > 0$. Now, for $s > \alpha$,

$$
L[g(t)u(t-a)](s) = L[f(t-a)u(t-a)](s) = e^{-as} L[f(t)](s) = e^{-as} L[g(t+a)](s),
$$

where the second equality follows from Proposition 7.38.

**Example 7.40.** Determine the Laplace transform of $f(t) = 5t^3 u(t-6)$. 

![Figure 3. The function $u(t-a)$.](image-url)
Solution. Here we have \( g(t)u(t - a) \) with \( g(t) = 5t^3 \) and \( a = 6 \). Thus
\[
g(t + a) = g(t + 6) = 5(t + 6)^3,
\]
and using (19) in the corollary gives
\[
\mathcal{L}[5t^3u(t - 6)](s) = e^{-6s}\mathcal{L}[5(t + 6)^3](s) = 5e^{-6s}\mathcal{L}[t^3 + 18t^2 + 108t + 216](s)
\]
Now we have
\[
\mathcal{L}[5t^3u(t - 6)](s) = 5e^{-6s}\left( \frac{3!}{s^4} + 18 \cdot \frac{2!}{s^3} + 108 \cdot \frac{1!}{s^2} + 216 \cdot \frac{1}{s} \right)
\]
\[
= \left( \frac{30}{s^4} + \frac{180}{s^3} + \frac{540}{s^2} + \frac{1080}{s} \right)e^{-6s},
\]
using linearity and Table 7.1.

Example 7.41. Determine the inverse Laplace transform of
\[
G(s) = \frac{e^{-2s}}{s^2 + 9}.
\]
Solution. Let \( f(t) \) be the function for which \( \mathcal{L}[f(t)](s) = 1/(s^2 + 9) \). Setting \( a = 2 \) in (18) gives
\[
\mathcal{L}[f(t - 2)u(t - 2)](s) = e^{-2s}F(s) = G(s).
\]
From Table 7.1 we find that
\[
\mathcal{L}\left[ \frac{1}{3} \sin 3t \right](s) = \frac{1}{3} \cdot \frac{3}{s^2 + 9} = \frac{1}{s^2 + 9} = \mathcal{L}[f(t)](s),
\]
so \( f(t) = \frac{1}{3} \sin 3t \) and from (20) we obtain
\[
\mathcal{L}\left[ \frac{1}{3} \sin(3t - 6)u(t - 2) \right](s) = G(s).
\]
Therefore
\[
\mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2 + 9} \right](t) = \frac{1}{3} \sin(3t - 6)u(t - 2).
\]

Definition 7.42. Let \( -\infty < a < b < \infty \). The **rectangular window function** associated with \( a \) and \( b \) is the function \( \Pi_{a,b} \) given by
\[
\Pi_{a,b}(t) = u(t - a) - u(t - b) = \begin{cases} 
0, & \text{if } t < a \\
1, & \text{if } a \leq t < b \\
0, & \text{if } t \geq b
\end{cases}
\]
The unit step function and rectangular window function can be employed to characterize piecewise-defined functions using a single expression.
Example 7.43. To express the function

\[ f(t) = \begin{cases} 
3t^2, & \text{if } t < -2 \\
0, & \text{if } -2 \leq t < 1 \\
2t, & \text{if } 1 \leq t < 3 \\
t \sin t, & \text{if } t \geq 3 
\end{cases} \]

in terms of the functions \( u \) and \( \Pi \), note that

\[ 3t^2 - 3t^2u(t + 2) = \begin{cases} 
3t^2, & \text{if } t < -2 \\
0, & \text{if } t \geq -2 
\end{cases} \]

and

\[ 2t\Pi_{1,3}(t) = \begin{cases} 
0, & \text{if } t < 1 \\
2t, & \text{if } 1 \leq t < 3 \\
0, & \text{if } t \geq 3 
\end{cases} \]

and

\[ tu(t - 3) \sin t = \begin{cases} 
0, & \text{if } t < 3 \\
t \sin t, & \text{if } t \geq 3 
\end{cases} \]

Summing these three functions gives the desired result:

\[ f(t) = 3t^2 - 3t^2u(t + 2) + 2t\Pi_{1,3}(t) + tu(t - 3) \sin t. \]

Notice in particular that \( f(-2) = 3(-2)^2 - 3(-2)^2 + 0 + 0 = 0 \), as required.

If it’s desired that \( f \) be expressed exclusively in terms of \( u \), simply observe that \( \Pi_{1,3}(t) = u(t - 1) - u(t - 3) \), and so

\[ f(t) = 3t^2 - 3t^2u(t + 2) + 2t[u(t - 1) - u(t - 3)] + tu(t - 3) \sin t, \]

or equivalently

\[ f(t) = 3t^2 - 3t^2u(t + 2) + 2tu(t - 1) + (t \sin t - 2t)u(t - 3). \]

Example 7.44. Find the Laplace transform of the function \( f \) in Example 7.43.

Solution. From the final expression for \( f(t) \) obtain above, we have

\[
\mathcal{L}[f](s) = \mathcal{L}[3t^2 - 3t^2u(t + 2) + 2t[u(t - 1) - u(t - 3)] + tu(t - 3) \sin t](s)
\]

\[
= 3\mathcal{L}[t^2](s) - 3\mathcal{L}[t^2u(t + 2)](s) + 2\mathcal{L}[tu(t - 1)](s) + \mathcal{L}[t \sin t \cdot u(t - 3)](s)
\]

\[
- 2\mathcal{L}[tu(t - 3)](s).
\]

We now use Table 7.I and (19) to obtain

\[
\mathcal{L}[f](s) = 3 \cdot \frac{2!}{s^{2+1}} - 3e^{2s}\mathcal{L}[(t - 2)^2](s) + 2e^{-s}\mathcal{L}[t + 1](s) + e^{-3s}\mathcal{L}[(t + 3) \sin(t + 3)](s)
\]

\[
- 2e^{-3s}\mathcal{L}[t + 3](s).
\]
To determine \( \mathcal{L}[(t - 2)^2](s) \) we simply expand the polynomial,
\[
\mathcal{L}[(t - 2)^2](s) = \mathcal{L}[t^2 - 4t + 4](s) = \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}.
\]
As for \( \mathcal{L}[(t + 3) \sin(t + 3)](s) \), the trigonometric identity \( \sin(u + v) = \sin u \cos v + \cos u \sin v \) will prove useful, giving
\[
\mathcal{L}[(t + 3) \sin(t + 3)](s) = \mathcal{L}[(t + 3)(\sin t \cos 3 + \cos t \sin 3)](s)
= (\cos 3)\mathcal{L}[t \sin t](s) + (\sin 3)\mathcal{L}[t \cos t](s) + (3 \cos 3)\mathcal{L}[\sin t](s)
+ (3 \sin 3)\mathcal{L}[\cos t](s)
= \frac{2s \cos 3}{(s^2 + 1)^2} + \frac{(s^2 - 1) \sin 3}{(s^2 + 1)^2} + \frac{3 \cos 3}{s^2 + 1} + \frac{3 \sin 3}{s^2 + 1}.
\]
Gathering all our results, we have
\[
\mathcal{L}[f](s) = \frac{6}{s^3} - 3e^{2s} \left( \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s} \right) + 2e^{-s} \left( \frac{1}{s^2} + \frac{1}{s} \right) + \frac{2s \cos 3 + (s^2 - 1) \sin 3}{(s^2 + 1)^2} + \frac{3 \cos 3 + 3s \sin 3}{s^2 + 1} - 2e^{-3s} \left( \frac{1}{s^2} + \frac{3}{s} \right),
\]
certainly no trivial expression!

**Example 7.45.** Solve the initial value problem \( y'' - y = f(t), \ y(0) = 1, \ y'(0) = 2, \) where \( f \) is given by
\[
f(t) = \begin{cases} 
1, & \text{if } 0 \leq t < 3 \\
t, & \text{if } t \geq 3 
\end{cases}
\]

**Solution.** We start by expressing \( f \) in terms of \( u \):
\[
f(t) = 1 + (-1 + t)u(t - 3).
\]
Now we have \( y'' - y = 1 - u(t - 3) + tu(t - 3) \). Taking the Laplace transform of each side, linearity properties yield
\[
\mathcal{L}[y''](s) - \mathcal{L}[y](s) = \mathcal{L}[1](s) - \mathcal{L}[u(t - 3)](s) + \mathcal{L}[tu(t - 3)](s).
\]
Now, letting \( Y(s) = \mathcal{L}[y](s) \), by \( \text{(10)} \) and \( \text{(19)} \) we have
\[
s^2Y(s) - sy(0) - y'(0) - Y(s) = \frac{1}{s} - \frac{e^{-3s}}{s} + e^{-3s}\mathcal{L}[t + 3](s).
\]
Using the given initial conditions then leads to
\[
s^2Y(s) - s - 2 - Y(s) = \frac{1}{s} - \frac{e^{-3s}}{s} + e^{-3s} \left( \frac{1}{s^2} + \frac{3}{s} \right) = \frac{1}{s} + e^{-3s} \left( \frac{1}{s^2} + \frac{2}{s} \right).
\]
So the function \( Y \) can be seen to be given by
\[
Y(s) = \frac{1}{s^2 - 1} \left( s + 2 + \frac{1}{s} + \frac{2s + 1}{s^2} e^{-3s} \right) = \frac{s + 2}{s^2 - 1} + \frac{1}{s(s^2 - 1)} + \frac{2s + 1}{s^2(s^2 - 1)} e^{-3s}.
\]
Partial fraction decomposition on the rightmost expression yields
\[
Y(s) = \left( \frac{3/2}{s - 1} - \frac{1/2}{s + 1} \right) + \left( -\frac{1}{s} + \frac{1/2}{s - 1} + \frac{1/2}{s + 1} \right) + \left( -\frac{2}{s^2} + \frac{3/2}{s - 1} + \frac{1/2}{s + 1} \right) e^{-3s},
\]
which a little algebra renders as
\[
Y(s) = \frac{2}{s - 1} - \frac{1}{s} - \frac{2}{s} e^{-3s} - \frac{1}{s^2} e^{-3s} + \frac{3/2}{s - 1} e^{-3s} - \frac{1/2}{s + 1} e^{-3s}.
\]
Hence
\[
y(t) = 2\mathcal{L}^{-1}\left[ \frac{1}{s - 1} \right](t) - \mathcal{L}^{-1}\left[ \frac{1}{s} \right](t) - 2\mathcal{L}^{-1}\left[ \frac{1}{s} e^{-3s} \right](t) - \mathcal{L}^{-1}\left[ \frac{1}{s^2} e^{-3s} \right](t) + \frac{3}{2}\mathcal{L}^{-1}\left[ \frac{1}{s - 1} e^{-3s} \right](t) - \frac{1}{2}\mathcal{L}^{-1}\left[ \frac{1}{s + 1} e^{-3s} \right](t).
\]
By Table 7.1 and (18), then,
\[
y(t) = 2e^t - 1 - 2u(t - 3) - (t - 3)u(t - 3) + \frac{3}{2} e^{t-3}u(t - 3) - \frac{1}{2} e^{3-t}u(t - 3).
\]
Therefore
\[
y(t) = 2e^t - 1 + \left( 1 - t + \frac{3}{2} e^{t-3} - \frac{1}{2} e^{3-t} \right) u(t - 3)
\]
is the solution to the initial value problem.

The solution to the IVP in Example 7.45 is seen to be
\[
y(t) = \begin{cases} 
2e^t - 1, & 0 \leq t < 3 \\
2e^t - t + \frac{3}{2} e^{t-3} + \frac{1}{2} e^{3-t}, & t \geq 3
\end{cases}
\]
Note that the graph of \( y(t) \), shown in Figure 4, does not exhibit any manifestly unusual properties at \( t = 3 \) or anywhere else! In fact the smooth appearance of the graph at \( t = 3 \) should lead us to wonder whether \( y(t) \) is differentiable there despite being piecewise-defined. We have
\[
y'_+(3) = \lim_{t \to 3^+} \frac{y(t) - y(3)}{t - 3} = \lim_{t \to 3^+} \frac{(2e^t - t + \frac{3}{2} e^{t-3} + \frac{1}{2} e^{3-t}) - (2e^3 - 1)}{t - 3},
\]

\[\text{Figure 4.}\]
which has a 0/0 indeterminate form and so by L'Hôpital's Rule it follows that
\[ y_+'(3) \overset{LR}{=} \lim_{t \to 3^+} \left( 2e^t - 1 + \frac{3}{2} e^{t-3} - \frac{1}{2} e^{3-t} \right) = 2e^3. \]
In similar fashion we obtain
\[ y_-'(3) = \lim_{t \to 3^-} \frac{y(t) - y(3)}{t - 3} = \lim_{t \to 3^-} \frac{(2e^t - 1) - (2e^3 - 1)}{t - 3} \overset{LR}{=} \lim_{t \to 3^-} 2e^t = 2e^3. \]
We see that \( y(t) \) is differentiable, and thus continuous, at \( t = 3 \) with \( y'(3) = 2e^3 \), and we have
\[ y'(t) = \begin{cases} 
2e^t, & 0 \leq t < 3 \\
2e^t - 1 + \frac{3}{2} e^{t-3} - \frac{1}{2} e^{3-t}, & t \geq 3 
\end{cases} \]
(Since \( \text{Dom}(y) = [0, \infty) \), at \( t = 0 \) there is strictly speaking a right-hand derivative \( y'_+(0) \) only.) Moreover
\[ \lim_{t \to 3} y'(t) = 2e^3 = y'(3) \]
shows that \( y'(t) \) is continuous at \( t = 3 \) as well.

Now we investigate the second derivative \( y''(t) \):
\[ y_+''(3) = \lim_{t \to 3^+} \frac{y'(t) - y'(3)}{t - 3} = \lim_{t \to 3^+} \frac{(2e^t - 1 + \frac{3}{2} e^{t-3} - \frac{1}{2} e^{3-t}) - 2e^3}{t - 3} \]
\[ \overset{LR}{=} \left( 2e^t + \frac{3}{2} e^{t-3} + \frac{1}{2} e^{3-t} \right) = 2e^3 + 2 \]
and
\[ y_-''(3) = \lim_{t \to 3^-} \frac{y'(t) - y'(3)}{t - 3} = \lim_{t \to 3^-} \frac{2e^t - 2e^3}{t - 3} \overset{LR}{=} \lim_{t \to 3^-} 2e^t = 2e^3 \]
Since \( y_+''(3) \neq y_-''(3) \) we conclude that \( y''(3) \) does not exist and so \( y''(t) \) is not differentiable at \( t = 3 \). Indeed \( y''(t) \) has a jump discontinuity of \( +2 \) in value at \( t = 3 \) precisely as \( f(t) \) on the right-hand side of the ODE does. We have
\[ y''(t) = \begin{cases} 
2e^t, & 0 \leq t < 3 \\
2e^t + \frac{3}{2} e^{t-3} + \frac{1}{2} e^{3-t}, & t > 3 
\end{cases} \]
Because \( 3 \not\in \text{Dom}(y'') \) must we accept that \( y(t) \) is not a solution to the IVP on \([0, \infty)\), but “only” on \([0, 3) \cup (3, \infty)\)? There are a few options. One option is the route of the engineer or physicist: \( t = 3 \) is merely an instant in time, so we refrain from considering what is happening to the physical system modeled by the IVP during that instant. Another option is to let \( y_+''(3) \) stand in for the value of \( y''(t) \) at \( t = 3 \), since the result does indeed satisfy the ODE:
\[ y_+''(3) - y(3) = f(3) \Rightarrow (2e^3 + 2) - (2e^3 - 1) = 3 \Rightarrow 3 = 3. \]
A third option adopted by some textbooks (which is really the first option writ large) is to use a version of the unit step function \( u(t) \) that is not defined at \( t = 0 \), so that \( f(t) \) given by (21) is not defined at \( t = 3 \) and we are relieved at the outset of any expectation to come up with a solution to the IVP there.\(^3\)

\(^3\)This third option we do not entertain for reasons mentioned at the beginning of this section.
Many physical phenomena are modeled by a differential equation of the form
\[ a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(t), \]
where the nonhomogeneity \( f \) is such that
\[ f(t) = \begin{cases} M, & t_0 - \epsilon < t < t_0 + \epsilon \\ 0, & \text{otherwise} \end{cases} \]
for some large \( M > 0 \) and small \( \epsilon > 0 \). Such a function is called an impulse function, which typically is constant in value on the short interval \((t_0 - \epsilon, t_0 + \epsilon)\) where it is nonzero, although it is not a requirement. The total impulse of \( f \), which could represent a force, voltage, or some other physical quantity that varies as a function of time \( t \), is defined to be
\[ I(f) = \int_{-\infty}^{\infty} f(t) \, dt = \int_{t_0-\epsilon}^{t_0+\epsilon} f(t) \, dt. \]
In particular, setting \( t_0 = 0 \), we may have
\[ f(t) = d_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon}, & |t| < \epsilon \\ 0, & |t| \geq \epsilon \end{cases} \]
in which case
\[ I(d_\epsilon) = \int_{-\infty}^{\infty} d_\epsilon(t) \, dt = \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon} \, dt = \frac{1}{2\epsilon} [(\epsilon) - (-\epsilon)] = 1 \]
for any \( \epsilon > 0 \). Observe that the smaller \( \epsilon \) becomes (i.e. the shorter the time the impulse occurs), the larger \( \frac{1}{2\epsilon} \) becomes (i.e. the greater the magnitude of the impulse), with the net effect being a total impulse of 1. The function \( d_\epsilon \) is called a unit impulse function.

As \( \epsilon \) tends to zero, we find that \( d_\epsilon \) approaches a kind of idealized unit impulse function that occurs “instantaneously” at \( t = 0 \) and has “infinite” magnitude. We have
\[ \lim_{\epsilon \to 0^+} d_\epsilon(t) = 0 \] (22)
for all \( t \neq 0 \), and also
\[ \lim_{\epsilon \to 0^+} I(d_\epsilon) = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} d_\epsilon(t) \, dt = \lim_{\epsilon \to 0^+} \int_{-\epsilon}^{\epsilon} d_\epsilon(t) \, dt = \lim_{\epsilon \to 0^+} (1) = 1. \] (23)

Equations (22) and (23) serve as motivation for the following definition.

**Definition 7.46.** The Dirac delta is the idealized unit impulse function \( \delta \) given by \( \delta(t) = 0 \) for all \( t \neq 0 \), and with the formal property
\[ \int_{-\infty}^{\infty} \delta(t) \, dt = 1. \] (24)

The Dirac delta is not a function in the conventional sense. No conventional function \( f \) can be zero everywhere except at one point, and yet manage to have a nonzero proper or improper Riemann integral \( \int_{a}^{b} f \) for some choice of limits \( a \) and \( b \). Rigorous justification of the Dirac delta is beyond the scope of this text. For our purposes the Dirac delta is a formal device.
that enables us to conveniently—and accurately—model physical systems involving impulse functions. If \( t_0 \neq 0 \), then an immediate consequence of Definition 7.46 is that

\[
\delta(t - t_0) = 0, \quad t \neq t_0,
\]

and

\[
\int_{-\infty}^{\infty} \delta(t - t_0) \, dt = 1.
\]

Since

\[
d_\epsilon(t - t_0) = \begin{cases} 
\frac{1}{2\epsilon}, & t_0 - \epsilon < t < t_0 + \epsilon \\
0, & t \leq t_0 - \epsilon \text{ or } t \geq t_0 + \epsilon
\end{cases}
\]

we see from (25) that

\[
\delta(t - t_0) = \lim_{\epsilon \to 0^+} d_\epsilon(t - t_0),
\]

which motives yet another formal definition.

**Definition 7.47.** For \( t_0 > 0 \) we define

\[
\mathcal{L}[\delta(t - t_0)](s) = \lim_{\epsilon \to 0^+} \mathcal{L}[d_\epsilon(t - t_0)](s).
\]

**Theorem 7.48.** If \( t_0 > 0 \), then

\[
\mathcal{L}[\delta(t - t_0)](s) = e^{-st_0}.
\]

**Proof.** Let \( t_0 > 0 \). Then there exists \( \epsilon > 0 \) sufficiently small that \( t_0 - \epsilon > 0 \), and so

\[
\mathcal{L}[d_\epsilon(t - t_0)](s) = \int_0^\infty e^{-st}d_\epsilon(t - t_0) \, dt = \int_{t_0-\epsilon}^{t_0+\epsilon} \frac{e^{-st}}{2\epsilon} \, dt
\]

\[
= \frac{1}{2\epsilon} \left[ -\frac{1}{s} e^{-st} \right]_{t_0-\epsilon}^{t_0+\epsilon} = -\frac{1}{2\epsilon s} \left[ e^{-s(t_0+\epsilon)} - e^{-s(t_0-\epsilon)} \right].
\]

Now, by Definition 7.47

\[
\mathcal{L}[\delta(t - t_0)](s) = \lim_{\epsilon \to 0^+} \mathcal{L}[d_\epsilon(t - t_0)](s) = \lim_{\epsilon \to 0^+} \frac{e^{-st_0}}{2} \left( \frac{e^{s\epsilon} - e^{-s\epsilon}}{s\epsilon} \right),
\]

and since the limit at right has indeterminate form \( 0/0 \) we may apply L’Hôpital’s Rule (differentiating with respect to \( \epsilon \)) to obtain

\[
\mathcal{L}[\delta(t - t_0)](s) = \lim_{\epsilon \to 0^+} \frac{e^{-st_0}}{2} \left( \frac{s e^{s\epsilon} + s e^{-s\epsilon}}{s} \right) = \lim_{\epsilon \to 0^+} \frac{e^{-st_0}}{2} \left( e^{s\epsilon} + e^{-s\epsilon} \right)
\]

\[
= \frac{e^{-st_0}}{2} (e^0 + e^0) = e^{-st_0},
\]

as was to be shown.

We can extend the result of Theorem 7.48 to the case when \( t_0 = 0 \) with a natural definition:

\[
\mathcal{L}[\delta(t)](s) := \lim_{t_0 \to 0} e^{-st_0} = 1
\]

for all \( s \in [0, \infty) \).

Generalizing the spirit of Definition 7.47 we have the following.
Definition 7.49. If $f$ is a continuous function, then
\[ \int_{-\infty}^{\infty} f(t) \delta(t - t_0) \, dt = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} f(t) d_\epsilon(t - t_0) \, dt \]
for any $t_0 \in \mathbb{R}$.

Theorem 7.50. If $f$ is continuous and $t_0 \in \mathbb{R}$, then
\[ \int_{-\infty}^{\infty} f(t) \delta(t - t_0) \, dt = f(t_0). \] (27)

Proof. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $t_0 \in \mathbb{R}$. We have
\[ \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} f(t) d_\epsilon(t - t_0) \, dt = \lim_{\epsilon \to 0^+} \int_{t_0-\epsilon}^{t_0+\epsilon} f(t) \cdot \frac{1}{2\epsilon} \, dt = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \int_{t_0-\epsilon}^{t_0+\epsilon} f(t) \, dt. \] (28)

Now, by the Mean Value Theorem for Integrals, there exists some $t^*_\epsilon \in (t_0 - \epsilon, t_0 + \epsilon)$, which depends on $\epsilon$, such that
\[ f(t^*_\epsilon) = \frac{1}{(t_0 + \epsilon) - (t_0 - \epsilon)} \int_{t_0-\epsilon}^{t_0+\epsilon} f(t) \, dt, \]
and thus
\[ \int_{t_0-\epsilon}^{t_0+\epsilon} f(t) \, dt = 2\epsilon f(t^*_\epsilon). \]

Returning to (28),
\[ \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} f(t) d_\epsilon(t - t_0) \, dt = \lim_{\epsilon \to 0^+} \left( \frac{1}{2\epsilon} \cdot 2\epsilon f(t^*_\epsilon) \right) = \lim_{\epsilon \to 0^+} f(t^*_\epsilon). \]

Let $\alpha > 0$ be arbitrary. Since $f$ is continuous at $t_0$, there exists some $\beta > 0$ such that
\[ |t - t_0| < \beta \implies |f(t) - f(t_0)| < \alpha. \]

Suppose that $\epsilon > 0$ is such that $\epsilon < \beta$. Then
\[ t^*_\epsilon \in (t_0 - \epsilon, t_0 + \epsilon) \subseteq (t_0 - \beta, t_0 + \beta), \]
which is to say $|t^*_\epsilon - t_0| < \beta$ and so
\[ |f(t^*_\epsilon) - f(t_0)| < \alpha. \]

This shows that
\[ \lim_{\epsilon \to 0^+} f(t^*_\epsilon) = f(t_0), \]
and therefore
\[ \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} f(t) d_\epsilon(t - t_0) \, dt = f(t_0). \]

Now (27) follows by Definition 7.49. ■

Example 7.51. Solve the initial value problem
\[ y'' + 4y = 2\delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0. \]

\footnote{See §6.1 of [CAL]}
Solution. We take the Laplace transform of each side of the ODE, using linearity properties to obtain
\[ \mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = 2\mathcal{L}[\delta(t - \pi)](s) - \mathcal{L}[\delta(t - 2\pi)](s). \]

Now, letting \( Y(s) = \mathcal{L}[y](s) \) and using Theorem 7.48, we obtain
\[ [s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = 2e^{-\pi s} - e^{-2\pi s}, \]
whence
\[ Y(s) = \frac{2e^{-\pi s}}{s^2 + 4} - \frac{e^{-2\pi s}}{s^2 + 4} \]
and so
\[ y(t) = \mathcal{L}^{-1}\left[ \frac{2e^{-\pi s}}{s^2 + 4} \right](t) - \mathcal{L}^{-1}\left[ \frac{e^{-2\pi s}}{s^2 + 4} \right](t). \] (29)

If we define \( h(t) = \sin(2t) \), then
\[ \mathcal{L}[h(t)](s) = \frac{2}{s^2 + 4}, \]
so by Proposition 7.38
\[ \mathcal{L}[h(t - \pi)u(t - \pi)](s) = e^{-\pi s}\mathcal{L}[h(t)](s) = \frac{2e^{-\pi s}}{s^2 + 4} \]
and hence
\[ \mathcal{L}^{-1}\left[ \frac{2e^{-\pi s}}{s^2 + 4} \right](t) = h(t - \pi)u(t - \pi) = \sin(2t - 2\pi)u(t - \pi) = \sin(2t)u(t - \pi). \]

In similar fashion we obtain
\[ \mathcal{L}^{-1}\left[ \frac{e^{-2\pi s}}{s^2 + 4} \right](t) = \frac{1}{2}h(t - 2\pi)u(t - 2\pi) = \frac{1}{2}\sin(2t - 4\pi)u(t - 2\pi) = \frac{1}{2}\sin(2t)u(t - 2\pi). \]

Putting these results into (29) yields
\[ y(t) = \sin(2t)\left[ u(t - \pi) - \frac{1}{2}u(t - 2\pi) \right], \]

\[ \text{Figure 5.} \]
or equivalently

\[ y(t) = \begin{cases} 
0, & 0 \leq t < \pi \\
\sin(2t), & \pi \leq t < 2\pi \\
\frac{1}{2}\sin(2t), & t \geq 2\pi 
\end{cases} \]

See Figure 5 for the graph of \( y(t) \).

**Example 7.52.** Solve the initial value problem

\[ y'' + y' + 2y = 5\delta(t - 3), \quad y(0) = 0, \quad y'(0) = 1. \]

**Solution.** We take the Laplace transform of each side of the ODE, using linearity properties to obtain

\[ \mathcal{L}[y''](s) + \mathcal{L}[y'](s) + 2\mathcal{L}[y](s) = 5\mathcal{L}[\delta(t - 3)](s). \]

Letting \( Y(s) = \mathcal{L}[y](s) \) and using Theorem 7.48, we obtain

\[ [s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] + 2Y(s) = 5e^{-3s}, \]

whence

\[ [s^2Y(s) - 1] + sY(s) + 2Y(s) = 5e^{-3s} \Rightarrow Y(s) = \frac{1 + 5e^{-3s}}{s^2 + s + 2}. \]

Since \( s^2 + s + 2 \) is an irreducible quadratic, we cast it as a sum of squares:

\[ s^2 + s + 2 = \left[s^2 + s + \left(\frac{1}{2}\right)^2\right] + 2 - \left(\frac{1}{2}\right)^2 = (s + \frac{1}{2})^2 + \left(\frac{\sqrt{7}}{2}\right)^2. \]

From

\[ Y(s) = \frac{1}{(s + 1/2)^2 + 7/4} + \frac{5e^{-3s}}{(s + 1/2)^2 + 7/4} \]

we obtain

\[ y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s + 1/2)^2 + 7/4}\right](t) + 5\mathcal{L}^{-1}\left[\frac{e^{-3s}}{(s + 1/2)^2 + 7/4}\right](t). \tag{30} \]

Referring to Table 1, we find that

\[ \mathcal{L}^{-1}\left[\frac{1}{(s + 1/2)^2 + 7/4}\right](t) = \frac{2}{\sqrt{7}}\mathcal{L}^{-1}\left[\frac{\sqrt{7}/2}{(s + 1/2)^2 + (\sqrt{7}/2)^2}\right](t) = \frac{2}{\sqrt{7}}e^{-t/2}\sin(\sqrt{7}t/2). \]

Now, if we let

\[ h(t) = \frac{2}{\sqrt{7}}e^{-t/2}\sin(\sqrt{7}t/2), \]

then by Proposition 7.38

\[ \mathcal{L}[h(t - 3)u(t - 3)](s) = e^{-3s}\mathcal{L}[h(t)](s) = \frac{e^{-3s}}{(s + 1/2)^2 + 7/4} \]

and thus

\[ \mathcal{L}^{-1}\left[\frac{e^{-3s}}{(s + 1/2)^2 + 7/4}\right](t) = h(t - 3)u(t - 3) = \frac{2}{\sqrt{7}}e^{-(t-3)/2}\sin(\sqrt{7}(t - 3)/2)u(t - 3). \]
Putting these results into (30) yields
\[ y(t) = \frac{2}{\sqrt{7}} e^{-t/2} \sin(\sqrt{7}t/2) + \frac{10}{\sqrt{7}} e^{-(t-3)/2} \sin(\sqrt{7}(t-3)/2)u(t-3), \]
the graph of which is presented in Figure 6. ■

Example 7.53. Solve the initial value problem
\[ y'' + y = \delta(t - 2\pi) \cos(t), \quad y(0) = 0, \quad y'(0) = 1. \]

Solution. We take the Laplace transform of each side of the ODE, using linearity properties and letting \( Y(s) = \mathcal{L}[y(t)](s) \) to obtain
\[ [s^2Y(s) - sy(0) - y'(0)] + Y(s) = \mathcal{L}[\delta(t - 2\pi) \cos(t)](s). \] (31)
Since \( \delta(t - 2\pi) = 0 \) for all \( t < 0 \), we have
\[ \int_0^\infty e^{-st}\delta(t - 2\pi) \cos(t) \, dt = \int_{-\infty}^{\infty} e^{-st}\delta(t - 2\pi) \cos(t) \, dt, \]
and so by Theorem 7.50
\[ \mathcal{L}[\delta(t - 2\pi) \cos(t)](s) = \int_{-\infty}^{\infty} e^{-st}\delta(t - 2\pi) \cos(t) \, dt = e^{-2\pi s} \cos(2\pi) = e^{-2\pi s}. \]
Equation (31) now becomes
\[ [s^2Y(s) - 1] + Y(s) = e^{-2\pi s}, \]
whence
\[ Y(s) = \frac{1}{s^2 + 1} + \frac{e^{-2\pi s}}{s^2 + 1}, \]
and finally
\[ y(t) = \mathcal{L}^{-1}\left[ \frac{1}{s^2 + 1} \right](t) + \mathcal{L}^{-1}\left[ \frac{e^{-2\pi s}}{s^2 + 1} \right](t). \]
With Table 1 and Proposition 7.38 we obtain
\[ y(t) = \sin(t) + \sin(t - 2\pi)u(t - 2\pi), \]
or simply

\[ y(t) = \sin(t)[1 + u(t - 2\pi)] \]

as the solution to the IVP.
Definition 7.54. Let \( f, g \) be piecewise continuous on \([0, \infty)\). The convolution of \( f \) and \( g \) is the function \( f \ast g : [0, \infty) \to \mathbb{R} \) given by

\[
(f \ast g)(t) = \int_0^t f(t - \tau)g(\tau) \, d\tau.
\]

Proposition 7.55. If \( f, g, h \) are piecewise continuous on \([0, \infty)\), then

1. \( f \ast g = g \ast f \)
2. \( f \ast (g + h) = f \ast g + f \ast h \)
3. \( f \ast (g \ast h) = (f \ast g) \ast h \)
4. \( f \ast 0 = 0 \)

Theorem 7.56 (Convolution Theorem). If \( f, g \) are piecewise continuous on \([0, \infty)\) and of exponential order \( \alpha \), then

\[
\mathcal{L}[f \ast g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s).
\]

for all \( s > \alpha \).

If we set \( F(s) = \mathcal{L}[f](s) \) and \( G(s) = \mathcal{L}[g](s) \), then

\[
\mathcal{L}[f \ast g](s) = F(s)G(s) \Rightarrow \mathcal{L}^{-1}[F(s)G(s)](t) = (f \ast g)(t)
\]

derives from the Convolution Theorem.

Example 7.57. Use the Convolution Theorem to find the inverse Laplace transform of

\[
H(s) = \frac{s}{(s^2 + 1)^2}.
\]

Solution. We have

\[
\mathcal{L}^{-1}[H(s)](t) = \mathcal{L}^{-1}\left[\frac{s}{(s^2 + 1)^2}\right](t) = \mathcal{L}^{-1}\left[\frac{s}{s^2 + 1} \cdot \frac{1}{s^2 + 1}\right](t) = \mathcal{L}^{-1}[F(s)G(s)](t)
\]

where

\[
F(s) = \frac{s}{s^2 + 1} \quad \text{and} \quad G(s) = \frac{1}{s^2 + 1}.
\]

Letting \( f(t) = \cos t \) and \( g(t) = \sin t \), we readily see that \( F(s) = \mathcal{L}[f](s) \) and \( G(s) = \mathcal{L}[g](s) \), and therefore

\[
\mathcal{L}^{-1}[H(s)](t) = \mathcal{L}^{-1}[F(s)G(s)](t) = (f \ast g)(t) = (\cos \ast \sin)(t).
\]

That is,

\[
\mathcal{L}^{-1}[H(s)](t) = (\cos \ast \sin)(t) = \int_0^t \cos(t - \tau) \sin(\tau) \, d\tau,
\]

and so using the trigonometric identity

\[
\sin x \cos y = \frac{1}{2}[\sin(x + y) + \sin(x - y)]
\]
we obtain
\[
\mathcal{L}^{-1}[H(s)](t) = \frac{1}{2} \int_0^t [\sin t + \sin(2\tau - t)] \, d\tau = \frac{1}{2} \left[ \tau \sin t - \frac{1}{2} \cos(2\tau - t) \right]_0^t
\]
\[
= \frac{1}{2} \left[ (t \sin t - \frac{1}{2} \cos t) - \left( 0 - \frac{1}{2} \cos(-t) \right) \right] = \frac{1}{2} t \sin t.
\]

\textbf{Example 7.58.} Solve the integral equation
\[
y(t) + \int_0^t (t - \tau)^2 y(\tau) \, d\tau = t^3 + 3.
\]

\textbf{Solution.} We have
\[
\int_0^t (t - \tau)^2 y(\tau) \, d\tau = (f \ast g)(t)
\]
with \( f(t) = t^2 \) and \( g(t) = y(t) \), so the integral equation may be written as
\[
y(t) + (f \ast y)(t) = t^3 + 3.
\]
Taking the Laplace transform of both sides of the equation yields, by the Convolution Theorem,
\[
\mathcal{L}[y](s) + \mathcal{L}[f](s)\mathcal{L}[y](s) = \mathcal{L}[t^3](s) + \mathcal{L}[3](s),
\]
or
\[
Y(s) + Y(s)L[t^2](s) = \mathcal{L}[t^3](s) + \mathcal{L}[3](s)
\]
if we let \( Y(s) = \mathcal{L}[y](s) \). Using a table of Laplace transforms yields
\[
Y(s) + Y(s) \cdot \frac{2}{s^3} = \frac{6}{s^4} + \frac{3}{s} \quad \Rightarrow \quad Y(s) \left( \frac{2 + s^3}{s^4} \right) = \frac{3(2 + s^3)}{s^4} \quad \Rightarrow \quad Y(s) = \frac{3}{s},
\]
whence
\[
y(t) = \mathcal{L}^{-1} \left[ \frac{3}{s} \right](t) = 3
\]

obtains as the (unique) solution.
### A Table of Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$\mathcal{L}<a href="s">f</a>$</th>
<th>Dom($\mathcal{L}[f]$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t \sin bt$</td>
<td>$\frac{2bs}{(s^2 + b^2)^2}$</td>
<td>$s &gt; 0$</td>
</tr>
<tr>
<td>$t \cos bt$</td>
<td>$\frac{s^2 - b^2}{(s^2 + b^2)^2}$</td>
<td>$s &gt; 0$</td>
</tr>
<tr>
<td>$e^{at} \sin bt$</td>
<td>$\frac{b}{(s - a)^2 + b^2}$</td>
<td>$s &gt; a$</td>
</tr>
<tr>
<td>$e^{at} \cos bt$</td>
<td>$\frac{s - a}{(s - a)^2 + b^2}$</td>
<td>$s &gt; a$</td>
</tr>
<tr>
<td>$e^{at}t^n$, $n = 0, 1, \ldots$</td>
<td>$\frac{n!}{(s - a)^{n+1}}$</td>
<td>$s &gt; a$</td>
</tr>
<tr>
<td>$u(t - a)$, $a \geq 0$</td>
<td>$\frac{e^{-as}}{s}$</td>
<td>$s &gt; 0$</td>
</tr>
<tr>
<td>$\delta(t - a)$, $a \geq 0$</td>
<td>$e^{-as}$</td>
<td>$s &gt; 0$</td>
</tr>
<tr>
<td>$(f * g)(t)$</td>
<td>$\mathcal{L}<a href="s">f(t)</a>\mathcal{L}<a href="s">g(t)</a>$</td>
<td>$s &gt; 0$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{t}}$</td>
<td>$\sqrt{\frac{\pi}{s}}$</td>
<td>$s &gt; 0$</td>
</tr>
<tr>
<td>$\sqrt{t}$</td>
<td>$\frac{1}{2s} \sqrt{\frac{\pi}{s}}$</td>
<td>$s &gt; 0$</td>
</tr>
<tr>
<td>$t^{n-1/2}$, $n = 1, 2, \ldots$</td>
<td>$\frac{1 \cdot 3 \cdot 5 \cdots (2n - 1) \sqrt{\pi}}{2^n s^{n+1/2}}$</td>
<td>$s &gt; 0$</td>
</tr>
</tbody>
</table>