# Math 242 Exam #3 Key (Summer 2023)

**1** Set $g(x,y) = 2x^2 + 3xy + 2y^2 - 7$, so the constraint is $g(x,y) = 0$. Find all $(x,y) \in \mathbb{R}^2$ for which there can be found some $\lambda \in \mathbb{R}$ such that the system

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y,z) \\ f_y(x,y) = \lambda g_y(x,y,z) \\ g(x,y) = 0 \end{cases}$$

has a solution. Explicitly the system is

$$\begin{cases} 2x = \lambda(4x+3y) \\ 2y = \lambda(3x+4y) \\ 7 = 2x^2 + 3xy + 2y^2 \end{cases} \tag{1}$$

The 1st equation gives $\lambda = \frac{2x}{4x+3y}$. Put this into the 2nd equation and manipulate to get $x^2 = y^2$, and thus $y = \pm x$. Putting $y = x$ into the 3rd equation and solving yields $x = \pm 1$, and so $(1,1)$ and $(-1,-1)$ are solutions to the system. Putting $y = -x$ into the 3rd equation and solving yields $x = \pm\sqrt{7}$, and so $(\sqrt{7}, -\sqrt{7})$ and $(-\sqrt{7}, \sqrt{7})$ are solutions to the system. Now evaluate $f$ at each of the four points found:

$$f(1,1) = f(-1,-1) = 2 \quad \text{and} \quad f(\sqrt{7}, -\sqrt{7}) = f(-\sqrt{7}, \sqrt{7}) = 14.$$

Thus $f$ attains a maximum value of 14 on the curve $C$ given by $g$, and a minimum value of 2.

**2** Making the substitution $u = \sin y$ along the way, we have

$$\begin{aligned} \int_0^{\pi/2} \int_0^{\cos y} e^{\sin y}\, dx\, dy &= \int_0^{\pi/2} \left[x e^{\sin y}\right]_0^{\cos y} dy = \int_0^{\pi/2} e^{\sin y} \cos y\, dy \\ &= \int_0^1 e^u\, du = [e^u]_0^1 = e - 1. \end{aligned}$$

**3** The order $dy\,dx$ works well, and integration by parts can be used to figure out $\int \tan x\, dx$:

$$\int_0^{\pi/3} \int_0^1 x \sec^2(xy)\, dy\, dx = \int_0^{\pi/3} \big[\tan xy\big]_0^1 dx = \int_0^{\pi/3} \tan x\, dx = \big[\ln|\sec x|\big]_0^{\pi/3} = \ln 2.$$

**4** We have

$$\int_1^2 \int_0^{x^{3/2}} \frac{2y}{\sqrt{x^4+1}}\, dy\, dx = \int_1^2 \left[\frac{y^2}{\sqrt{x^4+1}}\right]_0^{x^{3/2}} dx = \int_1^2 \frac{x^3}{\sqrt{x^4+1}}\, dx = \frac{1}{4} \int_2^{17} \frac{1}{\sqrt{u}}\, du = \frac{\sqrt{17} - \sqrt{2}}{2}.$$

**5** Converting to polar coordinates...

$$\int_{-\pi/2}^{\pi/2} \int_0^4 (16 - r^2) r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} 64\, d\theta = 64\pi.$$

**6** The height function is

$$h(x) = (4 - x - y) - (x^2 + y^2 - x - y) = 4 - x^2 - y^2,$$

while the region of integration $R$ will be the region in the $xy$-plane enclosed by the curve that is the projection onto $z = 0$ of the curve of intersection of the surfaces. The curve of intersection is $4 - x - y = x^2 + y^2 - x - y$, or $x^2 + y^2 = 4$, which is a circle with center $(0, 0)$ and radius 2, and so in polar coordinates

$$R = \{(r, \theta) : 0 \le \theta \le 2\pi,\ \ 0 \le r \le 2\}.$$

The volume of the solid is

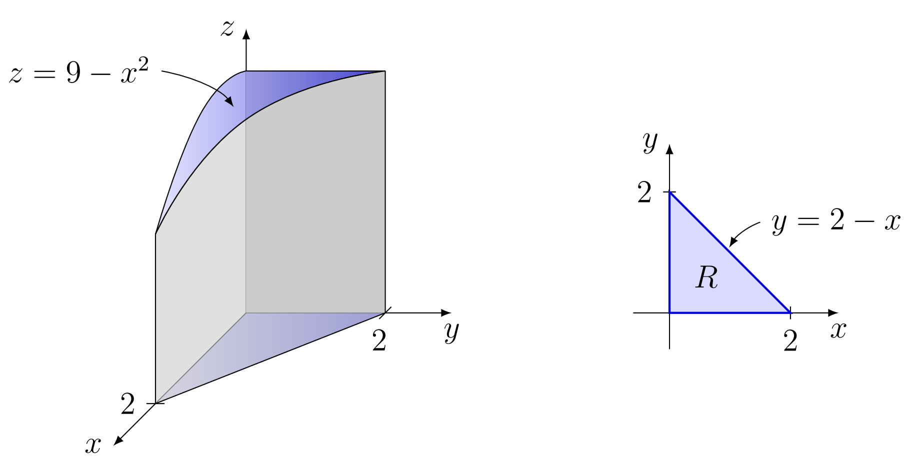
$$\mathcal{V} = \iint_R h = \int_0^{2\pi} \int_0^2 (4 - r^2) r\, dr\, d\theta = \int_0^{2\pi} \left[2r^2 - \frac{1}{4}r^4\right]_0^2 d\theta = 8\pi.$$

**7** For any $(x, y, z) \in D$ we have $0 \le z \le 9 - x^2$. We can evaluate $\iiint_D dV$ in the order $dz\,dy\,dx$ (other orders are possible). See the figure below.

To determine the limits of integration for $y$ and $x$, project $D$ onto the $xy$-plane to obtain the region $R$ shown at right in the figure. There it can be seen that if $(x, y) \in R$, then $0 \le y \le 2 - x$ for $0 \le x \le 2$, and so the limits of integration for $y$ will be 0 and $2 - x$, and the limits of integration for $x$ will be 0 and 2. We obtain

$$\begin{aligned}
\mathcal{V}(D) &= \iiint_D dV = \int_0^2 \int_0^{2-x} \int_0^{9-x^2} dz\,dy\,dx \\
&= \int_0^2 \int_0^{2-x} (9 - x^2)\, dy dx = \int_0^2 \left[9y - x^2 y\right]_0^{2-x} dx \\
&= \int_0^2 \left[9(2 - x) - x^2(2 - x)\right] dx = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{9}{2}x^2 + 18x\right]_0^2 = \frac{50}{3}.
\end{aligned}$$

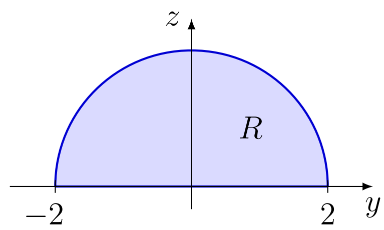
It can be instructive to try determining the volume of $D$ by integrating in the orders $dz\,dx\,dy$ and $dy\,dz\,dx$.




**8** On the $yz$-plane the region of integration is

$$R = \left\{(y, z) : 0 \le z \le \sqrt{4 - y^2},\ -2 \le y \le 2\right\},$$

the top half of a circular disc of radius 2:




This region is also expressible as

$$R = \{(y, z) : -\sqrt{4 - z^2} \le y \le \sqrt{4 - z^2},\ 0 \le z \le 2\},$$

and so the integral becomes

$$\int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\, dz\, dx.$$

To evaluate the integral let $z = 2\sin\theta$, so that $dz = 2\cos\theta\, d\theta$, and we obtain

$$\begin{aligned}\int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\, dz\, dx &= \int_0^1 \left( \int_0^{\pi/2} 2\sqrt{4 - 4\sin^2\theta} \cdot 2\cos\theta\, d\theta \right) dx \\ &= \int_0^1 \left( 8 \int_0^{\pi/2} \cos^2\theta\, d\theta \right) dx = 8 \int_0^1 \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\, d\theta\, dx \\ &= 8 \int_0^1 \left[ \frac{1}{2}\theta + \frac{\sin 2\theta}{4} \right]_0^{\pi/2} dx = 2\pi.\end{aligned}$$

**9** The cone and sphere intersect at $(x, y, z)$ where $x^2 + y^2 = z^2 = 2 - x^2 - y^2$, which is a curve in space that projects onto the $xy$-plane as the unit circle $x^2 + y^2 = 1$. In cylindrical coordinates the region of integration $D$ is thus

$$D = \{(r, \theta, z) : 0 \le \theta \le 2\pi,\ \ 0 \le r \le 1,\ \ r \le z \le \sqrt{2 - r^2}\}.$$

(Note that all $(r, \theta)$ such that $0 \le \theta \le 2\pi$ and $0 \le r \le 1$ covers the unit disc, whereas $z = r$ is the cone while $z = \sqrt{2 - r^2}$ is the sphere.) The volume is

$$V = \iiint_D dV = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} r\, dz\, dr\, d\theta = \frac{4\pi}{3}\left(\sqrt{2} - 1\right).$$

**10** In spherical coordinates the spheres are $\rho = 1$ and $\rho = 4$, and so the region $D$ is

$$D = \{(\rho, \varphi, \theta) : 0 \le \theta \le 2\pi,\ \ 0 \le \varphi \le \pi,\ \ 1 \le \rho \le 4\}.$$

Now,

$$\begin{aligned}\iiint_D (x^2 + y^2)\, dV &= \int_0^{2\pi} \int_0^{\pi} \int_1^4 \left[(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2\right] \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \int_1^4 \rho^4 \sin^3\varphi\, d\rho\, d\varphi\, d\theta = \frac{2728\pi}{5}.\end{aligned}$$