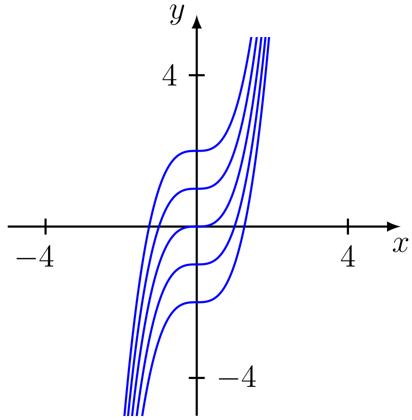
Math 242 Exam #2 Key (Summer 2023)

1 Dom $F = \mathbb{R}^2$ and Ran $F = [\ln 2, \ln 4]$.

2 We let $f(x, y) = c$ for various constants c, and graph $y = x^3 - c$. Letting $z = f(x, y)$, the level curves below are, from highest to lowest, $z = 2$, $z = 1$, $z = 0$, $z = -1$, $z = -2$.



3 Along the path $y = mx$ we get

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4} = \lim_{(x,mx)\to(0,0)} \frac{x^2(mx)^2}{x^4+(mx)^4} = \lim_{(x,mx)\to(0,0)} \frac{m^2}{1+m^4} = \frac{m^2}{1+m^4}.$$

So along path $y = 0$ the limit is 0, and along $y = x$ the limit is $\frac{1}{2}$. By the Two-Path Test the limit does not exist.

4 If L is the limit, then

$$L = \lim_{(x,y)\to(1,2)} \frac{\sqrt{y}-\sqrt{x+1}}{y-(x+1)} = \lim_{(x,y)\to(1,2)} \frac{\sqrt{y}-\sqrt{x+1}}{(\sqrt{y}-\sqrt{x+1})(\sqrt{y}+\sqrt{x+1})}$$

$$= \lim_{(x,y)\to(1,2)} \frac{1}{\sqrt{y}+\sqrt{x+1}} = \frac{1}{2\sqrt{2}}.$$

5a $\varphi_x = \dfrac{y^3}{1+(x^2y)^2} \cdot \dfrac{\partial}{\partial x}(x^2y) = \dfrac{2xy^4}{1+x^4y^2}$ and, with the product rule,

$$\varphi_y = 3y^2\tan^{-1}(x^2y) + \frac{y^3}{1+(x^2y)^2} \cdot \frac{\partial}{\partial y}(x^2y) = 3y^2\tan^{-1}(x^2y) + \frac{x^2y^3}{1+x^4y^2}.$$

5b $\psi_z = \dfrac{\partial}{\partial z}\left(\dfrac{y}{x+z}\right) = -\dfrac{y}{(x+z)^2}$ & $\psi_{xy} = \dfrac{\partial}{\partial y}\left(-\dfrac{y}{(x+z)^2}\right) = -\dfrac{1}{(x+z)^2}.$

6a Along the path $x = y^2$ the limit becomes

$$\lim_{(y^2,y)\to(0,0)} \frac{2y^4}{y^4+y^4} = \lim_{(y^2,y)\to(0,0)} (1) = 1,$$

which implies that

$$\lim_{(x,y)\to(0,0)} f(x,y) \neq 0 = f(0,0)$$

and therefore f is not continuous at $(0,0)$.

6b Function f is not differentiable at $(0,0)$ because it is not continuous there.

6c By definition we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0}(0) = 0,$$

and

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0}(0) = 0.$$

7 Here $w(t) = f(x,y)$ with $f(x,y) = \cos(2x)\sin(3y)$, $x = x(t) = t/2$ and $y = y(t) = t^4$. By the appropriate chain rule,

$$\begin{aligned} \frac{dw}{dt} &= f_x(x,y)x'(t) + f_y(x,y)y'(t) = -\sin(2x)\sin(3y) + 12t^3\cos(2x)\cos(3y) \\ &= -\sin(t)\sin(3t^4) + 12t^3\cos(t)\cos(3t^4). \end{aligned}$$

8a $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle -8x, -2y\rangle$.

8b $\dfrac{\nabla f(1,2)}{|\nabla f(1,2)|} = \dfrac{\langle -8,-4\rangle}{\sqrt{80}} = \dfrac{\langle -2,-1\rangle}{\sqrt{5}}$

8c $\pm\dfrac{\langle 1,-2\rangle}{\sqrt{5}}$

8d Let C_0 be given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \geq 0$. Then for any t the tangent vector to C_0 at the point $(x(t), y(t))$, which is $\mathbf{r}'(t)$, must be in the direction of $-\nabla f(x,y) = \langle 8x(t), 2y(t)\rangle$. Therefore we set

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \langle 8x(t), 2y(t)\rangle,$$

from which we obtain the differential equations $x' = 8x$ and $y' = 2y$. The method of separation of variables solves $x' = 8x$:

$$\frac{dx}{dt} = 8x \;\Rightarrow\; \int \frac{1}{8x}\,dx = \int dt \;\Rightarrow\; \frac{1}{8}\ln(8x) = t + K \;\Rightarrow\; x(t) = \frac{1}{8}e^{8t+8K} = Ae^{8t},$$

where K is an arbitrary constant, and $A = \frac{1}{8}e^{8K}$ is again arbitrary. Similarly $y' = 2y$ yields $y(t) = Be^{2t}$ for arbitrary B. Since C starts at $(1,2,4)$, we must have C_0 start at $(1,2)$, so $\mathbf{r}(0) = \langle x(0), y(0)\rangle = \langle 1,2\rangle$. Now, $A = Ae^0 = x(0) = 1$ and $B = Be^0 = y(0) = 2$, and so an equation for C_0 is

$$\mathbf{r}(t) = \langle e^{8t}, 2e^{2t}\rangle, \quad t \geq 0.$$

9 First get the unit vector in the direction of $\langle 5,12\rangle$, which is $\mathbf{u}=\frac{1}{13}\langle 5,12\rangle$. Now,

$$D_{\mathbf{u}}f(x,y)=\nabla f(x,y)\cdot\mathbf{u}=\langle 13ye^{xy},13xe^{xy}\rangle\cdot\frac{1}{13}\langle 5,12\rangle=\langle ye^{xy},xe^{xy}\rangle\cdot\langle 5,12\rangle=(12x+5y)e^{xy},$$

and so $D_{\mathbf{u}}f(1,0)=12$.

10a We have

$$f_x(x,y)=(y^2+xy+1)e^{xy}\quad\text{and}\quad f_y(x,y)=(x^2+xy+1)e^{xy}$$

Using

$$z=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+f(x_0,y_0)$$

with $(x_0,y_0)=(2,0)$, we get

$$z=f_x(2,0)(x-2)+f_y(2,0)(y-0)+f(2,0)=(x-2)+5y+2,$$

which simplifies to $x+5y-z=0$.

10b The tangent plane serves as a linearization L of the function f in a neighborhood of $(2,0)$, so that $z=f(x,y)\approx L(x,y)$ for (x,y) near $(2,0)$. From (1a) we have $z=x+5y$, so that

$$L(x,y)=x+5y,$$

and hence $z=f(1.95,0.05)\approx L(1.95,0.05)=1.95+5(0.05)=2.2$.

11 First we gather our partial derivatives:

$$\begin{aligned} f_x(x,y)&=-3x^2-6x\\ f_y(x,y)&=-3y^2+6y\\ f_{xx}(x,y)&=-6x-6\\ f_{yy}(x,y)&=-6y+6\\ f_{xy}(x,y)&=0 \end{aligned}$$

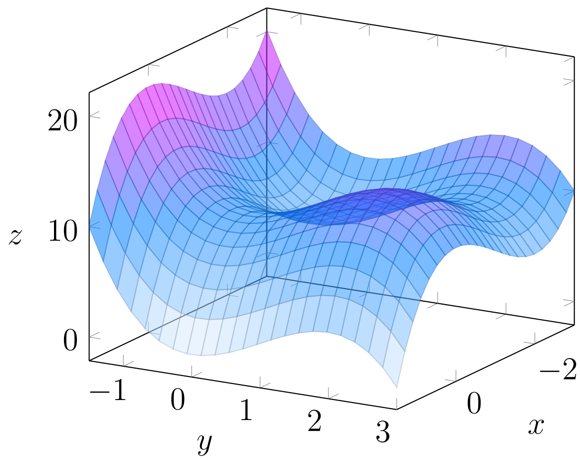
At no point does either f_x or f_y fail to exist, so we search for any point (x,y) for which $f_x(x,y)=f_y(x,y)=0$. This yields the system

$$\begin{cases} 3x^2+6x=0\\ 3y^2-6y=0 \end{cases}$$

The system has solutions $(0,0)$, $(0,2)$, $(-2,0)$, and $(-2,2)$. We construct a table:

(x,y)	f_{xx}	f_{yy}	f_{xy}	Φ	Conclusion
$(0,0)$	-6	6	0	-36	Saddle Point
$(0,2)$	-6	-6	0	36	Local Maximum
$(-2,0)$	6	6	0	36	Local Minimum
$(-2,2)$	6	-6	0	-36	Saddle Point

Below is a stereoscopic graph of a part of the surface containing the points of interest.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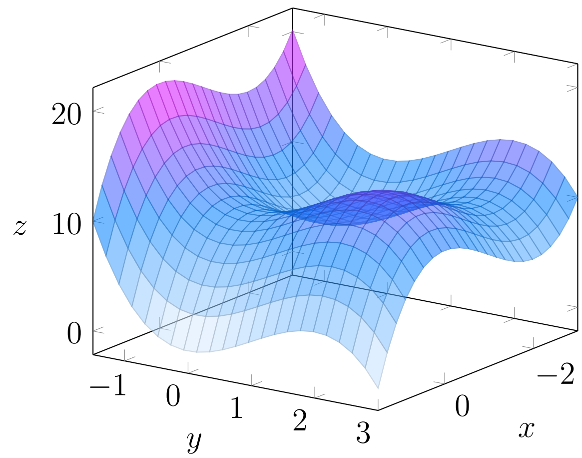
x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