Math 242 Exam #1 Key (Summer 2023)

1 Let $\mathbf{u} = \langle 4, -9\rangle$, so the vector is $\mathbf{v} = \dfrac{-12\mathbf{u}}{|\mathbf{u}|} = -\dfrac{12}{\sqrt{97}}\langle 4, -9\rangle$.

2 Using a special triangle from trigonometry is fastest: $\mathbf{v} = \dfrac{35}{2}\langle 1, \sqrt{3}\rangle$.

3 Completing squares yields

$$(x^2 - 6x + 9) + y^2 + (z^2 - 20z + 100) > -9 + 9 + 100 \quad \Rightarrow \quad (x-3)^2 + y^2 + (z-10)^2 > 100.$$

This is the region outside the closed ball of radius 10 centered at $(3, 0, 10)$.

4 First find a parametrization for the line containing the known points:

$$\mathbf{r}(t) = \langle 1,2,3\rangle + t\big(\langle 4,7,1\rangle - \langle 1,2,3\rangle\big) = \langle 1,2,3\rangle + t\langle 3,5,-2\rangle = \langle 1+3t, 2+5t, 3-2t\rangle.$$

Now, x and y must be such that $\mathbf{r}(t) = \langle x, y, 9\rangle$ for some t. That is,

$$\begin{cases} 1 + 3t = x \\ 2 + 5t = y \\ 3 - 2t = 9 \end{cases}$$

The last equation gives $t = -3$, and so we find that $x = -8$ and $y = -13$.

5a $|\mathbf{u}| = \sqrt{5}$ and $|\mathbf{v}| = \sqrt{89}$. Now,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{-10}{\sqrt{5}\sqrt{89}} \quad \Rightarrow \quad \theta = \cos^{-1}(0.4740) \approx 118.3^\circ.$$

5b We have

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = -\frac{10}{89}\langle -3, 8, 4\rangle.$$

6 $\mathbf{u} \perp \mathbf{v}$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$ if and only if $8 - 27 - c = 0$, and so $c = -19$ is required.

7 Suitable would be $\mathbf{v} = \langle 0,-1,2\rangle \times \langle 8,-2,-1\rangle = \langle 5, 16, 8\rangle$, but show the work

8 The direction vector $\mathbf{v}$ of the line is perpendicular to $\mathbf{j} = \langle 0,1,0\rangle$, so is parallel to $\mathbf{u}\times\mathbf{j}$. Thus we can let

$$\mathbf{v} = \mathbf{u}\times\mathbf{j} = \begin{vmatrix} 3 & -5 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -5 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix}\mathbf{k} = \langle 5, 0, 0\rangle,$$

and a parametrization is

$$\mathbf{r}(t) = \langle -3,-3,8\rangle + t\langle 5,0,0\rangle = \langle 5t-3, -3, 8\rangle, \quad t\in\mathbb{R}.$$

9 The direction vectors of the lines are $\langle 2,3,-1\rangle$ and $\langle -3,-4,-2\rangle$, which are not parallel vectors, and so the lines are not parallel. The lines intersect if and only if there is some s and t such that $\mathbf{r}(t)=\mathbf{R}(s)$. This gives us the system

$$\begin{cases} 5+2t=13-3s \\ 3+3t=13-4s \\ 1-\ \ t=\ \ 4-2s \end{cases}$$

The 3rd equation gives $t=2s-3$. Putting this into the 1st equation results in $s=2$, and hence $t=1$. However, putting $(s,t)=(2,1)$ into the 2nd equation results in $6=5$, so there is no solution to the system. The lines do not intersect, and therefore are skew.

10 We have

$$\mathbf{r}(t)=\left\langle \sin t+c_1,\ t+2e^{-t}+c_2,\ t-2e^{t}+c_3\right\rangle,$$

so $\mathbf{r}(0)=\langle c_1, 2+c_2, -2+c_3\rangle$. Since $\mathbf{r}(0)=\langle 1,1,1\rangle$, it follows that $c_1=1$, $c_2=-1$, $c_3=3$. Therefore

$$\mathbf{r}(t)=\left\langle \sin t+1,\ t+2e^{-t}-1,\ t-2e^{t}+3\right\rangle.$$

11 Let $p_0=(1,1,0)$, $p_1=(3,-1,4)$, $p_2=(1,2,3)$. Then

$$\mathbf{n}=\overrightarrow{p_0p_1}\times\overrightarrow{p_0p_2}=\langle 2,-2,4\rangle\times\langle 0,1,3\rangle=\langle -10,-6,2\rangle$$

is a normal vector for the plane. The equation of the plane is given by

$$\mathbf{n}\cdot(\langle x,y,z\rangle-\langle 1,1,0\rangle)=0,$$

or $-10x-6y+2z=-16$.

12 Find the solution set to the system

$$\begin{cases} x+2y-3z=1 \\ x+\ \ y+\ \ z=2 \end{cases}$$

The 2nd equation gives $z=2-x-y$, which when put into the 1st equation results in $4x+5y=7$, or $y=\frac{7}{5}-\frac{4}{5}x$, and hence $z=\frac{3}{5}-\frac{1}{5}x$. Solution set:

$$\left\{(x,\tfrac{7}{5}-\tfrac{4}{5}x,\tfrac{3}{5}-\tfrac{1}{5}x): x\in\mathbb{R}\right\}.$$

Replacing x with t, we parametrize the line as

$$\mathbf{r}(t)=\left\langle t,\tfrac{7}{5}-\tfrac{4}{5}t,\tfrac{3}{5}-\tfrac{1}{5}t\right\rangle,\quad t\in\mathbb{R}.$$

13 $\text{Dom}(\mathbf{r})=\{t\in\mathbb{R}: 4-t^2\ge 0,\ t\ge 0,\ 1+t>0\}$. So we must have $t\in[-2,2]$, $t\in[0,\infty)$, and $t\in(-1,\infty)$, which put together results in $t\in[0,2]$. Therefore $\text{Dom}(\mathbf{r})=[0,2]$.

14 $\mathbf{r}'(t)=\langle 9t^{7/2},3t^2\rangle$, and so $|\mathbf{r}'(t)|=\sqrt{81t^7+9t^4}=3t^2\sqrt{9t^3+1}$. Making the substitution $u=9t^3+1$ along the way, we find the length L of the curve to be

$$L=\int_0^3|\mathbf{r}'(t)|\,dt=\int_0^3 3t^2\sqrt{9t^3+1}\,dt=\int_1^{244}\frac{\sqrt{u}}{9}\,du=\frac{2}{27}(244^{3/2}-1).$$

15a First, $\mathbf{r}'(t) = \langle 1, t^{-1} \rangle$, so that $|\mathbf{r}'(t)| = \sqrt{1 + t^{-2}}$, and then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 1, t^{-1} \rangle}{\sqrt{1 + t^{-2}}} = \frac{\langle t, 1 \rangle}{\sqrt{t^2 + 1}}.$$

15b Using the quotient rule,

$$\mathbf{T}'(t) = \left\langle \frac{\sqrt{t^2+1} - t^2(t^2+1)^{-1/2}}{t^2+1}, -\frac{1}{2}(t^2+1)^{-3/2}(2t) \right\rangle = \frac{1}{(t^2+1)^{3/2}} \langle 1, -t \rangle,$$

and so the curvature is (after some simplications)

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{t}{(t^2+1)^{3/2}}.$$

15c Set $\kappa'(t) = 0$. Using the quotient rule and simplifying, we get

$$\frac{1 - 2t^2}{(t^2+1)^{5/2}} = 0,$$

so that $1 - 2t^2 = 0$. Solving for $t > 0$ yields $t = \frac{1}{\sqrt{2}}$. That is, the curvature is maximal when $t = \frac{1}{\sqrt{2}}$, and the maximum curvature value is $\kappa(\frac{1}{\sqrt{2}}) = \frac{2}{3\sqrt{3}}$ (which is about 0.385 but we want the exact value).