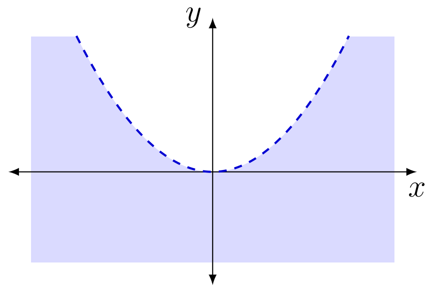
MATH 242 EXAM #2 KEY (SUMMER 2013)

1 The function h is a composition of a polynomial function and the natural logarithm function, and so it is continuous on its domain. We have

$$\text{Dom}(h) = \{(x,y) : x^2 - 3y > 0\} = \left\{(x,y) : y < \tfrac{1}{3}x^2\right\},$$

which is the shaded region in $\mathbb{R}^2$ illustrated below.



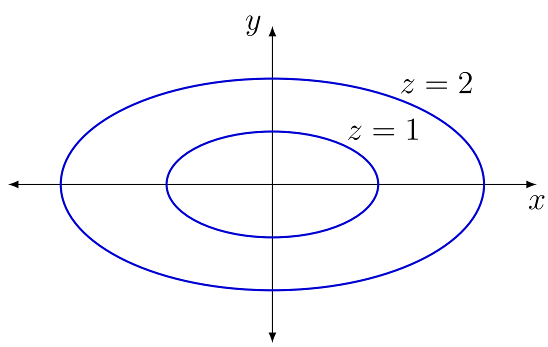
2 The level curve $z = 1$ has equation $1 = \sqrt{x^2 + 4y^2}$, which implies

$$x^2 + \frac{y^2}{1/4} = 1,$$

an ellipse. The level curve $z = 2$ has equation $2 = \sqrt{x^2 + 4y^2}$, which implies

$$\frac{x^2}{4} + y^2 = 1,$$

also an ellipse. Graph is below.



3 We have

$$\lim_{(x,y)\to(2,1)} \frac{x^2 - 4y^2}{x - 2y} = \lim_{(x,y)\to(2,1)} \frac{(x-2y)(x+2y)}{x - 2y} = \lim_{(x,y)\to(2,1)} (x + 2y) = 2 + 2(1) = 4.$$

4 First approach $(0,0)$ on the path $(x(t), y(t)) = (t, 0)$ (i.e. the x-axis), so the limit becomes:

$$\lim_{t\to 0} \frac{x(t)y(t) + y^3(t)}{x^2(t) + y^2(t)} = \lim_{t\to 0} \frac{0}{t^2 + 0} = 0.$$

Next, approach $(0,0)$ on the path $(x(t), y(t)) = (t,t)$ (i.e. the line $y = x$), so the limit becomes:

$$\lim_{t\to 0} \frac{x(t)y(t) + y^3(t)}{x^2(t) + y^2(t)} = \lim_{t\to 0} \frac{t^2 + t^3}{t^2 + t^2} = \lim_{t\to 0} \frac{t^2(1+t)}{2t^2} = \lim_{t\to 0} \frac{1+t}{2} = \frac{1}{2}.$$

The limits don't agree, so the original limit cannot exist by the Two-Path Test.

5a We have

$$g_x(x,y) = \ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} \quad \text{and} \quad g_y(x,y) = \frac{2xy}{x^2+y^2}.$$

5b We have

$$h_z(x,y,z) = -3\sin(x+2y+3z) \quad \text{and} \quad h_{zy}(x,y,z) = -6\cos(x+2y+3z).$$

6a Along the path $y = x$ the limit becomes

$$\lim_{(x,x)\to(0,0)} -\frac{x\cdot x}{x^2+x^2} = \lim_{(x,x)\to(0,0)} -\frac{1}{2} = -\frac{1}{2},$$

which implies that

$$\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 0$$

and therefore f is not continuous at $(0,0)$.

6b By an established theorem, since f is not continuous at $(0,0)$ it cannot be differentiable at $(0,0)$.

6c By definition we have

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0}(0) = 0.$$

Thus, even though f is not differentiable at $(0,0)$, it can have partial derivatives at $(0,0)$.

7 Here $w(t) = f(x,y)$ with $f(x,y) = \cos(2x)\sin(3y)$, $x = x(t) = t/2$ and $y = y(t) = t^4$. By Chain Rule 1 in notes,

$$\begin{aligned} w'(t) &= f_x(x,y)x'(t) + f_y(x,y)y'(t) = -\sin(2x)\sin(3y) + 12t^3\cos(2x)\cos(3y) \\ &= -\sin(t)\sin(3t^4) + 12t^3\cos(t)\cos(3t^4). \end{aligned}$$

8 Here $z(s,t) = f(x,y)$ with $f(x,y) = xy - 2x + 3y$, $x = x(s,t) = \sin s$ and $y = y(s,t) = \tan t$. By Chain Rule 2 in notes,

$$z_s(s,t) = f_x(x,y)x_s(s,t) + f_y(x,y)y_s(s,t) = (y-2)\cos s + (x+3)(0) = (\tan t - 2)\cos s,$$

and

$$z_t(s,t) = f_x(x,y)x_t(s,t) + f_y(x,y)y_t(s,t) = (y-2)(0) + (x+3)\sec^2 t = (\sin s + 3)\sec^2 t.$$

9a $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \langle -9x^2, 2\rangle$

9b Direction of steepest ascent is

$$\frac{\nabla f(1,2)}{|\nabla f(1,2)|} = \frac{\langle -9, 2\rangle}{\sqrt{(-9)^2 + 2^2}} = \frac{1}{\sqrt{85}}\langle -9, 2\rangle,$$

and direction of steepest descent is

$$-\frac{1}{\sqrt{85}}\langle -9, 2\rangle.$$

9c Let C_0 be given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \geq 0$. Then for any t the tangent vector to C_0 at the point $(x(t), y(t))$, which is $\mathbf{r}'(t)$, must be in the direction of $-\nabla f(x, y) = \langle 9x^2(t), -2\rangle$. Therefore we set

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \langle 9x^2(t), -2\rangle,$$

from which we obtain the differential equations $x' = 9x^2$ and $y' = -2$. The first equation can be solved by the Method of Separation of Variables:

$$\frac{dx}{dt} = 9x^2 \;\Rightarrow\; \frac{dx}{9x^2} = dt \;\Rightarrow\; \int \frac{1}{9x^2}\,dx = \int dt \;\Rightarrow\; -\frac{1}{9x} = t + K \;\Rightarrow\; x(t) = -\frac{1}{9t + K},$$

with arbitrary constant K. The equation $y' = -1$ easily gives $y(t) = -2t + K'$ for arbitrary constant K'. Since C is given to start at $(1, 2, 3)$, we must have C_0 start at $(1, 2)$; that is, $\mathbf{r}(0) = \langle x(0), y(0)\rangle = \langle 1, 2\rangle$. From $-1/(9 \cdot 0 + K) = x(0) = 1$ we obtain $K = -1$, and from $-2(0) + K' = y(0) = 2$ we obtain $K' = 2$. Therefore an equation for C_0 is

$$\mathbf{r}(t) = \left\langle \frac{1}{1 - 9t},\; 2 - 2t \right\rangle, \quad t \geq 0.$$

10 First get the unit vector in the direction of $\langle 1, \sqrt{3}\rangle$:

$$\mathbf{u} = \frac{\langle 1, \sqrt{3}\rangle}{2} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle.$$

Now,

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = \langle e^x \sin y, e^x \cos y\rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \frac{e^x \sin y}{2} + \frac{\sqrt{3} e^x \cos y}{2},$$

and so

$$D_{\mathbf{u}} f(0, \pi/4) = \frac{e^0 \sin(\pi/4)}{2} + \frac{\sqrt{3} e^0 \cos(\pi/4)}{2} = \frac{1/\sqrt{2}}{2} + \frac{\sqrt{3} \cdot 1/\sqrt{2}}{2} = \frac{\sqrt{2} + \sqrt{6}}{4}.$$

11a We have

$$f_x(x, y) = -\frac{2y}{(x - y)^2} \quad \text{and} \quad f_y(x, y) = \frac{2x}{(x - y)^2}$$

Using

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

with $(x_0, y_0) = (3, 2)$, we get

$$z = -4(x-3) + 6(y-2) + 5,$$

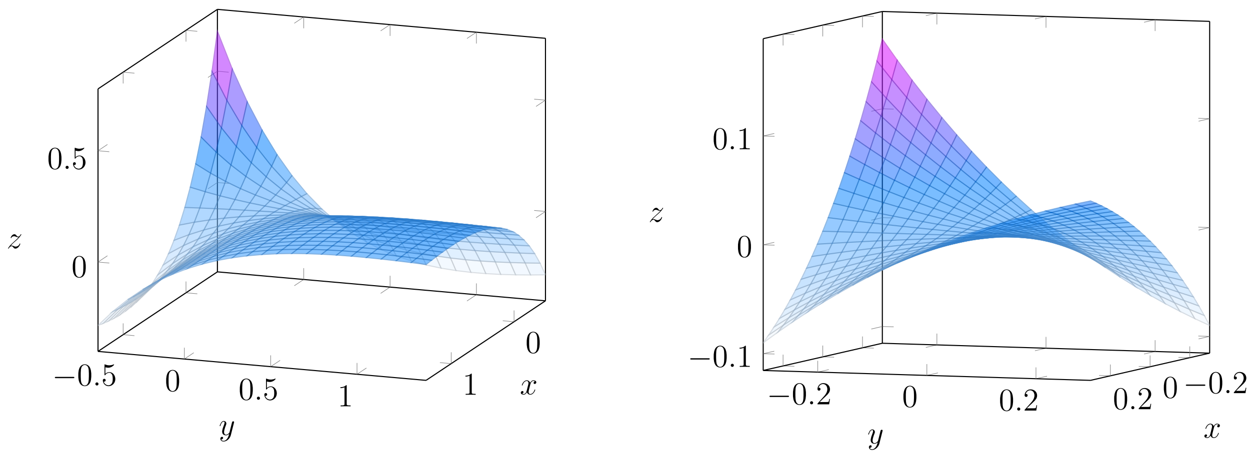
which simplifies to $4x - 6y + z = 5$.

11b The tangent plane serves as a linearization L of the function f in a neighborhood of $(3, 2)$, so that $L(x, y) \approx f(x, y)$ for (x, y) near $(3, 2)$. From (1a) we have

$$L(x, y) = -4x + 6y + 5,$$

and so $f(2.95, 2.05) \approx L(2.95, 2.05) = 5.5$.

12



First we gather our partial derivatives:

$$\begin{aligned} f_x(x,y) &= (y - xy)e^{-x-y} \\ f_y(x,y) &= (x - xy)e^{-x-y} \\ f_{xx}(x,y) &= (xy - 2y)e^{-x-y} \\ f_{yy}(x,y) &= (xy - 2x)e^{-x-y} \\ f_{xy}(x,y) &= (1 - x + xy - y)e^{-x-y} \end{aligned}$$

At no point does either f_x or f_y fail to exist, so we search for any point (x, y) for which $f_x(x, y) = f_y(x, y) = 0$. This yields the system

$$\begin{cases} y - xy = 0 \\ x - xy = 0 \end{cases}$$

We see we must have $x = xy = y$. Putting $x = y$ into the 1st equation yields $x - x^2 = 0$, which has solutions $x = 0, 1$. When $x = 0$ we obtain (from the 1st equation) $y = 0$; and when $x = 1$ we obtain (from the 2nd equation) $y = 1$. Thus we have solutions $(0, 0)$ and $(1, 1)$, which are critical points.

From $f_{xx}(0, 0) = f_{yy}(0, 0) = 0$ and $f_{xy}(0, 0) = 1$ we have $\Phi(0, 0) = -1 < 0$, and therefore f has a saddle point at $(0, 0)$ by the Second Derivative Test.

From $f_{xx}(1, 1) = f_{yy}(1, 1) = -e^{-2}$ and $f_{xy}(1, 1) = 0$ we have $\Phi(0, 0) = e^{-4} > 0$, and therefore f has a local maximum at $(1, 1)$ by the Second Derivative Test.

In the figure at left above, it is not at all obvious at a glance that there is a local maximum present, but it is there! The figure at right zooms in on $(0,0,0)$ to at least make the saddle point clear.

13 We have $f_x(x,y) = -2x$, $f_y(x,y) = -8y$, $f_{xx}(x,y) = -2$, $f_{yy}(x,y) = -8$, $f_{xy}(x,y) = 0$, and thus $\Phi(x,y) = 16$. Setting $f_x(x,y) = f_y(x,y) = 0$ yields the system $-2x = 0$ & $-8y = 0$, which gives $(0,0)$ as the only critical point, which is a point that lies in R. Since $f_{xx}(0,0) = -2 < 0$ and $\Phi(0,0) = 8 > 0$, f has a local maximum at $(0,0)$.

Along the top side of R we have $y = 1$, which yields the function $f_1(x) = 2 - x^2$ for $x \in [-2,2]$. Using the Closed Interval Method on f_1 in $[-2,2]$, the global maximum of f_1 occurs at $x = 0$ (corresponding to point $(0,1)$ for f), and the global minimum at $x = \pm 2$ (corresponding to points $(\pm 2, 1)$ for f).

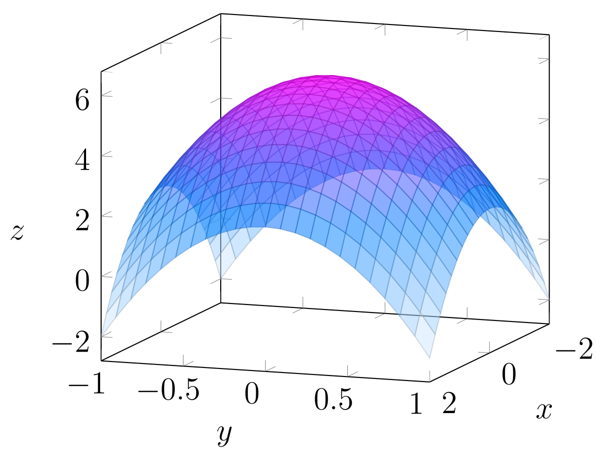
Along the bottom of R we have $y = -1$, which yields the function $f_2(x) = 2 - x^2$ for $x \in [-2,2]$. The global maximum of f_2 occurs at $x = 0$ (corresponding to point $(0,-1)$ for f), and the global minimum at $x = \pm 2$ (corresponding to points $(\pm 2, -1)$ for f).

Along the left side of R we have $x = -2$, which yields the function $f_3(y) = 2 - 4y^2$ for $y \in [-1,1]$. Using the Closed Interval Method on f_3 in $[-1,1]$, the global maximum of f_3 occurs at $y = 0$ (corresponding to point $(-2,0)$ for f), and the global minimum at $y = \pm 1$ (corresponding to points $(-2, \pm 1)$ for f).

Along the right side of R we have $x = 2$, which yields the function $f_4(y) = 2 - 4y^2$ for $y \in [-1,1]$. The global maximum of f_4 occurs at $y = 0$ (corresponding to point $(2,0)$ for f), and the global minimum at $y = \pm 1$ (corresponding to points $(2, \pm 1)$ for f).

Any point in R that corresponds to a point where any of the functions f_i has an extremum is a point where f itself has an extremum. Thus to find the global extrema of f we evaluate f at all these points as well as all critical points. We have: $f(\pm 2, \pm 1) = -2$, $f(0, \pm 1) = 2$, $f(\pm 2, 0) = 2$, and $f(0,0) = 6$.

Therefore f has a global minimum at the points $(\pm 2, \pm 1)$, and a global maximum at $(0,0)$.



14 By Fubini's Theorem we have

$$\iint_R e^{x+2y}\,dA = \int_1^{\ln 3}\int_0^{\ln 2} e^{x+2y}\,dxdy = \int_1^{\ln 3} e^{2y}\left(\int_0^{\ln 2} e^x\,dx\right)dy$$

$$= \int_1^{\ln 3} e^{2y} \left[e^x\right]_0^{\ln 2} dy = \int_1^{\ln 3} e^{2y}\, dy = \frac{1}{2}\left[e^{2y}\right]_1^{\ln 3} = \frac{1}{2}(9 - e^2) = \frac{9 - e^2}{2}.$$

15 By Fubini's Theorem we have

$$\iint_R y^3 \sin(xy^2)\, dA = \int_0^{\sqrt{\pi/2}} \int_0^1 y^3 \sin(xy^2)\, dxdy = \int_0^{\sqrt{\pi/2}} \left[-\frac{y^3}{y^2}\cos(xy^2)\right]_0^1 dy$$
$$= \int_0^{\sqrt{\pi/2}} -y(\cos y^2 - 1)\, dy = \int_0^{\sqrt{\pi/2}} y\, dy - \int_0^{\sqrt{\pi/2}} y\cos(y^2)\, dy$$
$$= \frac{\pi}{4} - \frac{1}{2}\int_0^{\sqrt{\pi/2}} \left[\sin(y^2)\right]' dy = \frac{\pi}{4} - \frac{1}{2}\left[\sin(y^2)\right]_0^{\sqrt{\pi/2}} = \frac{\pi}{4} - \frac{1}{2}.$$

16 In the first quadrant $y = x^2$ and $y = 8 - x^2$ intersect at $(2, 4)$, which allows us to determine R so that

$$\iint_R (x + y) dA = \int_0^2 \int_{x^2}^{8-x^2} (x + y) dydx = \int_0^2 \left[xy + \frac{1}{2}y^2\right]_{x^2}^{8-x^2} dx$$
$$= \int_0^2 \left[x(8 - x^2) + \frac{1}{2}(8 - x^2)^2 - x^3 - \frac{1}{2}x^4\right] dx$$
$$= \int_0^2 \left(32 + 8x - 8x^2 - 2x^3\right) dx = \frac{152}{3}.$$

17 The order $dydx$ will prove more tractable:

$$\int_0^{1/4} \int_0^{\sqrt{x}} y\cos(16\pi x^2) dydx = \int_0^{1/4} \left[\frac{y^2}{2}\cos(16\pi x^2)\right]_0^{\sqrt{x}} = \int_0^{1/4} \frac{x\cos(16\pi x^2)}{2}\, dx.$$

Now let $u = 16\pi x^2$ to obtain

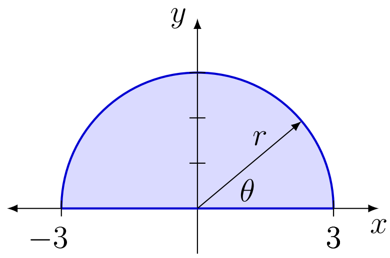
$$\int_0^{1/4} \frac{x\cos(16\pi x^2)}{2}\, dx = \int_0^{\pi} \frac{\cos u}{x} \cdot \frac{1}{32\pi}\, du = \frac{1}{64\pi}\int_0^{\pi} \cos u\, du = \frac{1}{64\pi}[\sin u]_0^{\pi} = 0.$$

18 The sketch of R in the xy-plane is below. The region

$$S = \{(r, \theta) : 0 \le r \le 3 \text{ and } 0 \le \theta \le \pi\}$$

in the $r\theta$-plane is such that $T_{\text{pol}}(S) = R$, and therefore

$$\iint_R 2xy\, dA = \iint_S 2(r\cos\theta)(r\sin\theta)r\, dA = \int_0^{\pi} \int_0^3 2(r\cos\theta)(r\sin\theta)r\, drd\theta$$
$$= \int_0^{\pi} \int_0^3 2r^3 \cos\theta \sin\theta\, drd\theta = \int_0^{\pi} \cos\theta\sin\theta \left[\frac{1}{2}r^4\right]_0^3 d\theta$$
$$= \frac{81}{2}\int_0^{\pi} \cos\theta\sin\theta\, d\theta = \frac{81}{4}\int_0^{\pi} \sin(2\theta)\, d\theta = 0.$$



19 By definition area is given by

$$\mathcal{A} = \int_0^{\pi} \int_0^{2\cos 3\theta} r\, dr d\theta = \int_0^{\pi} \left[\frac{1}{2} r^2\right]_0^{2\cos 3\theta} d\theta = 2\int_0^{\pi} \cos^2 3\theta\, d\theta$$
$$= \int_0^{\pi} \frac{1+\cos 6\theta}{2}\, d\theta = \int_0^{\pi} (1+\cos 6\theta) d\theta = \left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi} = \pi,$$

where along the way we make use of the old trigonometric identity

$$\cos^2 \alpha = \frac{1+\cos 2\alpha}{2}.$$

Note a critical thing: the entire curve is traced out exactly once as θ ranges from 0 to π, so if you integrate with respect to θ from 0 to 2π you will get the area times 2!