Math 242 Exam #1 Key (Summer 2013)

1 Two such vectors are

$$\frac{6\overrightarrow{qp}}{|\overrightarrow{qp}|} = \frac{6\langle -7, 8\rangle}{\sqrt{(-7)^2 + 8^2}} = \frac{6}{\sqrt{113}}\langle -7, 8\rangle$$

and

$$-\frac{6\overrightarrow{qp}}{|\overrightarrow{qp}|} = -\frac{6}{\sqrt{113}}\langle -7, 8\rangle$$

2 We have $\mathbf{F}_{PC} = \langle 200, 0\rangle$ and $\mathbf{F}_{GD} = \langle 0, -150\rangle$, and so the net force on the ATM is

$$\mathbf{F} = \mathbf{F}_{PC} + \mathbf{F}_{GD} = \langle 200, -150\rangle.$$

The magnitude of the force is

$$|\mathbf{F}| = \sqrt{200^2 + 150^2} = 250 \text{ N},$$

and the direction of the force is

$$\tan^{-1}(150/200) \approx 36.9^\circ \text{ south of east}$$

3 Canoe's velocity vector (relative to water) is $\mathbf{u} = \langle -8, 0\rangle$, while the water's velocity vector is

$$\mathbf{v} = \left\langle -3/\sqrt{2}, 3/\sqrt{2}\right\rangle.$$

The velocity of the canoe relative to the shore is thus

$$\mathbf{w} = \mathbf{u} + \mathbf{v} = \left\langle -8 - 3/\sqrt{2}, 3/\sqrt{2}\right\rangle.$$

The speed is thus $|\mathbf{w}| = 10.34$ km/h, and the direction is

$$\theta = \tan^{-1}\left(\frac{3/\sqrt{2}}{8 + 3/\sqrt{2}}\right) = \tan^{-1}(0.2326) = 11.8^\circ$$

north of west.

4 Midpoint is at $(-2, 2, 5)$, which is the center of the sphere. Letting $|p - q|$ denote the distance between p and q, radius of the sphere is

$$r = \frac{1}{2}|p - q| = \frac{1}{2}\sqrt{4^2 + 0^2 + 4^2} = \frac{1}{2}\sqrt{32} = 2\sqrt{2}.$$

Equation of the sphere is thus

$$(x + 2)^2 + (y - 2)^2 + (z - 5)^2 = 8.$$

5a $\|\mathbf{u}\| = \sqrt{2^2 + (-1)^2 + 5^2} = \sqrt{30}$ and $\|\mathbf{v}\| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}.$

5b Since $\mathbf{u}\cdot\mathbf{v} = (2)(-1) + (-1)(1) + (5)(1) = 2$ and $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 = (\sqrt{3}\,)^2 = 3$, we have

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{2}{3}\langle -1,1,1\rangle = \left\langle -\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle.$$

5c We have

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{2}{\sqrt{30}\sqrt{3}} = \frac{2}{3\sqrt{10}} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\frac{2}{3\sqrt{10}}\right) \approx 77.8^\circ.$$

6 The cross product of the two vectors will do:

$$\langle 6,-2,4\rangle \times \langle 1,2,3\rangle = -14\mathbf{i} - 14\mathbf{j} + 14\mathbf{k} = -14\langle 1,1,-1\rangle.$$

7 We have $y(t) = -t+4$, so $y(t) = -2$ implies that $-t+4 = -2$, or $t = 6$. So the line intersects the plane at the point

$$\mathbf{r}(6) = \big\langle 2(6)+1, -6+4, 6-6\big\rangle = \langle 13,-2,0\rangle.$$

8 Let $\mathbf{v} = \langle 3-1, -3-0, 3-1\rangle = \langle 2,-3,2\rangle$ and $\mathbf{r}_0 = \langle 1,0,1\rangle$. Then an equation (i.e. parameterization) for the line is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 1,0,1\rangle + \langle 2t,-3t,2t\rangle,$$

or

$$\mathbf{r}(t) = \langle 1+2t, -3t, 1+2t\rangle, \quad -\infty < t < \infty.$$

9 From $\mathbf{r}'(t) = \langle 1,0,-2/t^2\rangle$ we have

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1+4/t^4}}\left\langle 1,0,-\frac{2}{t^2}\right\rangle.$$

Now we obtain

$$\mathbf{T}(2) = \frac{1}{\sqrt{1+4/2^4}}\left\langle 1,0,-\frac{2}{2^2}\right\rangle = \left\langle \frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right\rangle$$

10 By definition,

$$\int_0^1 \langle e^{2t}, e^{-t}, t\rangle dt = \left\langle \int_0^1 e^{2t}dt, \int_0^1 e^{-t}dt, \int_0^1 t\,dt\right\rangle = \left\langle \left[\frac{1}{2}e^{2t}\right]_0^1, \left[-e^{-t}\right]_0^1, \left[\frac{1}{2}t^2\right]_0^1\right\rangle$$

$$= \left\langle \frac{e^2-1}{2},\ 1-\frac{1}{e},\ \frac{1}{2}\right\rangle.$$

11 From $\mathbf{a}(t) = \langle 1, t\rangle$ we integrate to obtain $\mathbf{v}(t) = \langle t + a_1, t^2/2 + a_2\rangle$. Now,

$$\mathbf{v}(0) = \langle a_1, a_2\rangle = \langle 2, -1\rangle,$$

so we have $a_1 = 2$ and $a_2 = -1$, and obtain

$$\mathbf{v}(t) = \langle t + 2, t^2/2 - 1\rangle$$

for the velocity function. Integrating this function then yields

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + 2t + b_1, \frac{1}{6}t^3 - t + b_2 \right\rangle.$$

Now,

$$\mathbf{r}(0) = \langle b_1, b_2\rangle = \langle 0, 8\rangle,$$

so $b_1 = 0$ and $b_2 = 8$ and we obtain

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + 2t, \frac{1}{6}t^3 - t + 8 \right\rangle.$$

12 First we obtain

$$\mathbf{r}'(t) = \langle -3\cos^2 t \sin t + 3\sin^2 t \cos t,$$

and then

$$\|\mathbf{r}'(t)\| = \sqrt{9\cos^4 t\cos^2 t + 9\sin^4 t\cos^2 t} = 3\sin t\cos t\sqrt{\cos^2 t + \sin^2 t} = 3\sin t\cos t.$$

Now, letting $u = \sin t$, we have

$$\mathcal{L}(C) = \int_0^{\pi/2} \|\mathbf{r}'(t)\| dt = 3\int_0^{\pi/2} \sin t\cos t\, dt = 3\int_0^1 u\, du = \frac{3}{2}.$$

13 The use of hyperbolic functions is not required to do this problem, but they do help streamline the process. We have

$$\mathbf{r}'(t) = \langle \sinh t, \cosh t, 1\rangle,$$

and thus

$$\|\mathbf{r}'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t,$$

using the hyperbolic identity $\cosh^2 t - \sinh^2 t = 1$ and recalling that $\cosh t > 0$ for all $t \in \mathbb{R}$. Now,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle \sinh t, \cosh t, 1\rangle}{\sqrt{2}\cosh t} = \frac{1}{\sqrt{2}}\langle \tanh t, 1, \operatorname{sech} t\rangle,$$

whence

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}\langle \operatorname{sech}^2 t, 0, -\tanh t\operatorname{sech} t\rangle = \frac{\operatorname{sech} t}{\sqrt{2}}\langle \operatorname{sech} t, 0, -\tanh t\rangle,$$

and finally

$$\|\mathbf{T}'(t)\| = \frac{\operatorname{sech} t}{\sqrt{2}}\sqrt{\operatorname{sech}^2 t + \tanh^2 t} = \frac{\operatorname{sech} t}{\sqrt{2}} = \frac{1}{\sqrt{2}\cosh t},$$

using the hyperbolic identity $\operatorname{sech}^2 t + \tanh^2 t = 1$. The curvature of C at $\mathbf{r}(t)$ is

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \left(\frac{1}{\sqrt{2}\cosh t}\right)\left(\frac{1}{\sqrt{2}\cosh t}\right) = \frac{1}{2\cosh^2 t} = \frac{\operatorname{sech}^2 t}{2},$$

or equivalently

$$\kappa(t) = \frac{2}{(e^t + e^{-t})^2}.$$

14a The vector function $\mathbf{r}(t) = \langle t, e^t \rangle$, $t \in (-\infty, \infty)$, yields the same curve. We have

$$\mathbf{r}'(t) = \langle 1, e^t \rangle \quad \text{and} \quad |\mathbf{r}'(t)| = \sqrt{1 + e^{2t}},$$

so that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1+e^{2t}}}\langle 1, e^t \rangle = \left\langle \frac{1}{\sqrt{1+e^{2t}}}, \frac{e^t}{\sqrt{1+e^{2t}}} \right\rangle,$$

which yields

$$\mathbf{T}'(t) = \left\langle -\frac{e^{2t}}{(1+e^{2t})^{3/2}}, \frac{e^t}{(1+e^{2t})^{3/2}} \right\rangle = \frac{e^t}{(1+e^{2t})^{3/2}}\langle -e^t, 1 \rangle,$$

and thus

$$|\mathbf{T}'(t)| = \frac{e^t}{(1+e^{2t})^{3/2}}\sqrt{e^{2t}+1} = \frac{e^t}{1+e^{2t}}.$$

Finally we obtain

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{e^t}{1+e^{2t}} \cdot \frac{1}{\sqrt{1+e^{2t}}} = \frac{e^t}{(1+e^{2t})^{3/2}}$$

as the curvature of the curve at the point (t, e^t).

14b To find the value of t for which $\kappa(t)$ attains a maximum value, we first find $\kappa'(t)$:

$$\kappa'(t) = \frac{(1+e^{2t})^{3/2}e^t - e^t \cdot \frac{3}{2}(1+e^{2t})^{1/2} \cdot 2e^{2t}}{(1+e^{2t})^3} = \frac{e^t - 2e^{3t}}{(1+e^{2t})^{5/2}}.$$

Now we set $\kappa'(t) = 0$ to obtain

$$\frac{e^t - 2e^{3t}}{(1+e^{2t})^{5/2}} = 0,$$

and hence

$$e^t - 2e^{3t} = 0.$$

From this comes the equation $e^{2t} = 1/2$, which has solution

$$t = \frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{\ln(2)}{2}.$$

Thus the curve has maximum curvature at the point

$$\mathbf{r}\left(-\frac{\ln(2)}{2}\right) = \left\langle -\frac{\ln(2)}{2}, e^{-\ln(2)/2} \right\rangle = \left\langle -\frac{\ln(2)}{2}, \frac{1}{\sqrt{2}} \right\rangle.$$

The value of the maximum curvature is

$$\kappa\left(-\tfrac{1}{2}\ln(2)\right) = \frac{e^{-\ln(2)/2}}{\left[1+e^{-\ln(2)}\right]^{3/2}} = \frac{1/\sqrt{2}}{(1+1/2)^{3/2}} = \frac{2\sqrt{3}}{9}.$$

15 For $p_0 = (1,1,0)$, $q_0 = (-2,8,4)$, $r_0 = (1,2,3)$, we have $\overrightarrow{p_0q_0} = \langle -3,7,4\rangle$ and $\overrightarrow{p_0r_0} = \langle 0,1,3\rangle$. Now,

$$\begin{aligned}\mathbf{n} = \overrightarrow{p_0q_0} \times \overrightarrow{p_0r_0} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 7 & 4 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 7 & 4 \\ 1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 4 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 7 \\ 0 & 1 \end{vmatrix}\mathbf{k} \\ &= 17\mathbf{i} + 9\mathbf{j} - 3\mathbf{k} = \langle 17, 9, -3\rangle,\end{aligned}$$

so if $p = (x,y,z)$, then the equation of the plane is given by

$$\mathbf{n} \cdot \overrightarrow{p_0p} = \langle 17,9,-3\rangle \cdot \langle x-1, y-1, z\rangle = 0,$$

or $17x + 9y - 3z = 26$.

16 We have planes $P : x + 2y - 3z = 1$ and $Q : x + y + z = 2$. Now, the intersection of P and the plane $z = 0$ is the set of points on the line $\ell_0 : x + 2y = 1$, and the intersection of Q and $z = 0$ is the line $\ell_0' : x + y = 2$. So the point that is an element of $\ell_0 \cap \ell_0'$ must be a point in $P \cap Q$. We find this point by finding the solution to the system

$$\begin{cases} x + 2y = 1 \\ x + y = 2 \end{cases}$$

which is $(3,-1)$. Thus $(3,-1,0) \in P \cap Q$ (since we're on the plane $z = 0$).

Next, the intersection of P and the plane $z = 1$ is the line $\ell_1 : x + 2y = 4$, and the intersection of Q and $z = 1$ is the line $\ell_1' : x + y = 1$. Again, a point in $\ell_1 \cap \ell_1'$ is a point in $P \cap Q$. The system

$$\begin{cases} x + 2y = 4 \\ x + y = 1 \end{cases}$$

has solution $(-2,3)$, and thus $(-2,3,1) \in P \cap Q$ (recall we're now on the plane where z is 1).

So the line of intersection for P and Q contains points $r_0(3,-1,0)$ and $r_1(-2,3,1)$. Let $\mathbf{v} = \overrightarrow{r_0r_1} = \langle -5,4,1\rangle$. An equation for the line is thus

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 3,-1,0\rangle + t\langle -5,4,1\rangle.$$