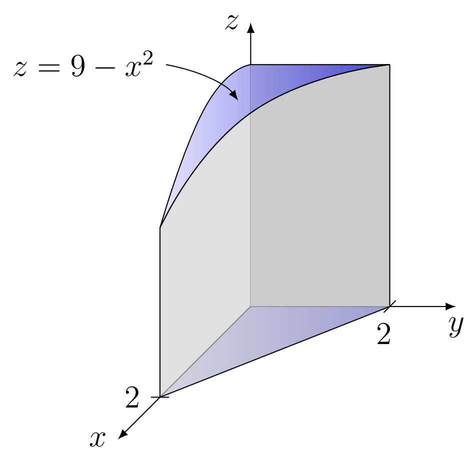
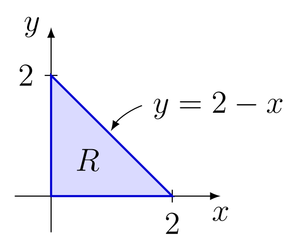
1 For any $(x, y, z) \in D$ we have $0 \le z \le 9 - x^2$. We can evaluate $\iiint_D dV$ in the order $dz\,dy\,dx$ (other orders are possible). See the figure below.

To determine the limits of integration for y and x, project D onto the xy-plane to obtain the region R shown at right in the figure. There it can be seen that if $(x, y) \in R$, then $0 \le y \le 2 - x$ for $0 \le x \le 2$, and so the limits of integration for y will be 0 and $2 - x$, and the limits of integration for x will be 0 and 2. We obtain

$$\begin{aligned}\mathcal{V}(D) &= \iiint_D dV = \int_0^2 \int_0^{2-x} \int_0^{9-x^2} dz\,dy\,dx \\ &= \int_0^2 \int_0^{2-x} (9 - x^2)\,dy dx = \int_0^2 \left[9y - x^2 y\right]_0^{2-x} dx \\ &= \int_0^2 \left[9(2 - x) - x^2(2 - x)\right] dx = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{9}{2}x^2 + 18x\right]_0^2 = \frac{50}{3}.\end{aligned}$$

It can be instructive to try determining the volume of D by integrating in the orders $dz\,dx\,dy$ and $dy\,dz\,dx$.

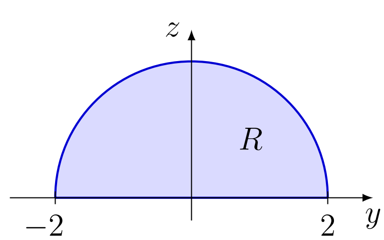




2 On the yz-plane the region of integration is

$$R = \left\{(y, z) : 0 \le z \le \sqrt{4 - y^2},\ -2 \le y \le 2\right\},$$

the top half of a circular disc of radius 2:



This region is also expressible as

$$R = \left\{(y, z) : -\sqrt{4 - z^2} \le y \le \sqrt{4 - z^2},\ 0 \le z \le 2\right\},$$

and so the integral becomes

$$\int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\, dz\, dx.$$

To evaluate the integral let $z = 2\sin\theta$, so that $dz = 2\cos\theta\, d\theta$, and we obtain

$$\begin{aligned}
\int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\, dz\, dx &= \int_0^1 \left(\int_0^{\pi/2} 2\sqrt{4 - 4\sin^2\theta} \cdot 2\cos\theta\, d\theta \right) dx \\
&= \int_0^1 \left(8 \int_0^{\pi/2} \cos^2\theta\, d\theta \right) dx = 8 \int_0^1 \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta\, dx \\
&= 8 \int_0^1 \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4} \right]_0^{\pi/2} dx = 2\pi.
\end{aligned}$$

3 The cone and sphere intersect at (x, y, z) where $x^2 + y^2 = z^2 = 2 - x^2 - y^2$, which is a curve in space that projects onto the xy-plane as the unit circle $x^2 + y^2 = 1$. In cylindrical coordinates the region of integration D is thus

$$D = \left\{ (r, \theta, z) : 0 \le \theta \le 2\pi,\ \ 0 \le r \le 1,\ \ r \le z \le \sqrt{2 - r^2} \right\}.$$

(Note that all (r, θ) such that $0 \le \theta \le 2\pi$ and $0 \le r \le 1$ covers the unit disc, whereas $z = r$ is the cone while $z = \sqrt{2 - r^2}$ is the sphere.) The volume is

$$V = \iiint_D dV = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} r\, dz\, dr\, d\theta = \frac{4\pi}{3}\left(\sqrt{2} - 1\right).$$

4 In spherical coordinates the spheres are $\rho = 1$ and $\rho = 4$, and so the region D is

$$D = \left\{ (\rho, \varphi, \theta) : 0 \le \theta \le 2\pi,\ \ 0 \le \varphi \le \pi,\ \ 1 \le \rho \le 4 \right\}.$$

Now,

$$\begin{aligned}
\iiint_D (x^2 + y^2)\, dV &= \int_0^{2\pi} \int_0^{\pi} \int_1^4 \left[(\rho \sin\varphi \cos\theta)^2 + (\rho \sin\varphi \sin\theta)^2 \right] \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \int_1^4 \rho^4 \sin^3\varphi\, d\rho\, d\varphi\, d\theta = \frac{2728\pi}{5}.
\end{aligned}$$

5 Parametrization for ellipse:

$$\mathbf{r}(t) = \langle 2\cos t, \sin t \rangle, \quad t \in [0, 2\pi).$$

This covers the ellipse precisely: for each point on the ellipse there is precisely one value of $t \in [0, 2\pi)$ such that $\mathbf{r}(t)$ is the position vector of that point. For $\mathbf{F}$ to be normal to the ellipse at the point $\mathbf{r}(t)$ means $\mathbf{r}'(t) \cdot \mathbf{F}(\mathbf{r}(t)) = 0$, which implies $\cos t \sin t = 0$. Solutions are $t = 0, \pi/2, \pi, 3\pi/2$, which correspond to points $(2, 0)$, $(0, 1)$, $(-2, 0)$, $(0, -1)$.

Points where $\mathbf{F}$ is tangential to the ellipse are points $\mathbf{r}(t)$ where $\mathbf{F}(\mathbf{r}(t))$ is parallel to $\mathbf{r}'(t)$; that is, there exists some constant $k \neq 0$ such that $\mathbf{r}'(t) = k\mathbf{F}(\mathbf{r}(t))$. This requires that

$$\langle -2\sin t, \cos t \rangle = k \langle 4\cos t, \sin t \rangle,$$

which is only satisfied if $4k^2 = -2$. There is no solution, so the vector field is never tangent to the ellipse.

6a A fine parametrization would be

$$\mathbf{r}(t) = \langle 0, -3, 2\rangle(1-t) + \langle 1, -7, 4\rangle t = \langle t, -4t-3, 2t+2\rangle, \quad t \in [0,1].$$

6b We have $\mathbf{r}'(t) = \langle 1, -4, 2\rangle$, so that $\|\mathbf{r}'(t)\| = \sqrt{21}$. Now,

$$\begin{aligned}\int_C (xz - y^2)\, ds &= \sqrt{21}\int_0^1 \left[t(2t+2) - (-4t-3)^2\right] dt \\ &= -\sqrt{21}\int_0^1 (14t^2 + 22t + 9)\, dt = -\frac{74\sqrt{21}}{3}.\end{aligned}$$

7 Making the substitution $u = t^2 - 1$ along the way, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^2 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\, dt = \int_0^2 \mathbf{F}(t^2, t^3)\cdot\langle 2t, 3t^2\rangle\, dt \\ &= \int_0^2 \langle e^{t^2-1}, t^5\rangle\cdot\langle 2t, 3t^2\rangle\, dt = \int_0^2 (2te^{t^2-1} + 3t^7)\, dt \\ &= \int_0^2 2te^{t^2-1}\, dt + \int_0^2 3t^7\, dt = \int_{-1}^3 e^u\, du + \frac{3}{8}\left[t^8\right]_0^2 = e^3 - \frac{1}{e} + 96.\end{aligned}$$

8 Here we have $x(t) = 2\cos t$ and $y(t) = 2\sin t$, so $x'(t) = -2\sin t$ and $y'(t) = 2\cos t$, and then

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{n} &= \int_0^{2\pi} \left[f(\mathbf{r}(t))y'(t) - g(\mathbf{r}(t))x'(t)\right] dt \\ &= \int_0^{2\pi} \left[f(2\cos t, 2\sin t)(2\cos t) - g(2\cos t, 2\sin t)(-2\sin t)\right] dt \\ &= \int_0^{2\pi} \left[(2\sin t - 2\cos t)(2\cos t) - (2\cos t)(-2\sin t)\right] dt \\ &= 4\int_0^{2\pi} 2\cos t\sin t\, dt - 4\int_0^{2\pi} \cos^2 t\, dt \\ &= \int_0^{2\pi} \sin(2t)\, dt - 4\int_0^{2\pi} \frac{1+\cos(2t)}{2}\, dt \\ &= 4\left[-\frac{1}{2}\cos(2t)\right]_0^{2\pi} - 2\left[t + \frac{1}{2}\sin(2t)\right]_0^{2\pi} \\ &= 4\cdot 0 - 2\cdot 2\pi = -4\pi.\end{aligned}$$

Thus there is a net flux of 4π *into* the region enclosed by C.