Math 242 Exam #3 Key (Fall 2018)

1a We have

$$f_x(x,y) = (y^2 + xy + 1)e^{xy} \quad \text{and} \quad f_y(x,y) = (x^2 + xy + 1)e^{xy}$$

Using

$$z = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) + f(x_0,y_0)$$

with $(x_0, y_0) = (2, 0)$, we get

$$z = f_x(2,0)(x-2) + f_y(2,0)(y-0) + f(2,0) = (x-2) + 5y + 2,$$

which simplifies to $x + 5y - z = 0$.

1b The tangent plane serves as a linearization L of the function f in a neighborhood of $(2, 0)$, so that $z = f(x,y) \approx L(x,y)$ for (x,y) near $(2,0)$. From (1a) we have $z = x + 5y$, so that

$$L(x,y) = x + 5y,$$

and hence $z = f(1.95, 0.05) \approx L(1.95, 0.05) = 1.95 + 5(0.05) = 2.2$.

2 S is given by $F(x,y,z) = 0$, where

$$F(x,y,z) = x^2 + y^2 - z^2 - 2x + 2y + 3.$$

So $F_x(x,y,z) = 2x - 2$, $F_y(x,y,z) = 2y + 2$, and $F_z(x,y,z) = -2z$. A tangent plane to S at $(a,b,c) \in S$ is given by

$$\nabla F \cdot \langle x-a, y-b, z-c\rangle = 0 \;\Rightarrow\; \langle 2a-2, 2b+2, -2c\rangle \cdot \langle x-a, y-b, z-c\rangle = 0,$$

which becomes

$$(a-1)x + (b+1)y - cz = a(a-1) + b(b+1) - c^2.$$

A horizontal plane is a plane with equation $z = k$, where k is some constant. Thus we need $a = 1$ and $b = -1$. Then

$$a^2 + b^2 - c^2 - 2a + 2b + 3 = 0 \;\Rightarrow\; c^2 = 1 \;\Rightarrow\; c = \pm 1.$$

Therefore the two points on S where the tangent plane is horizontal are $(1,-1,1)$ and $(1,-1,-1)$.

3 First we gather our partial derivatives:

$$\begin{aligned} f_x(x,y) &= -3x^2 - 6x \\ f_y(x,y) &= -3y^2 + 6y \\ f_{xx}(x,y) &= -6x - 6 \\ f_{yy}(x,y) &= -6y + 6 \\ f_{xy}(x,y) &= 0 \end{aligned}$$

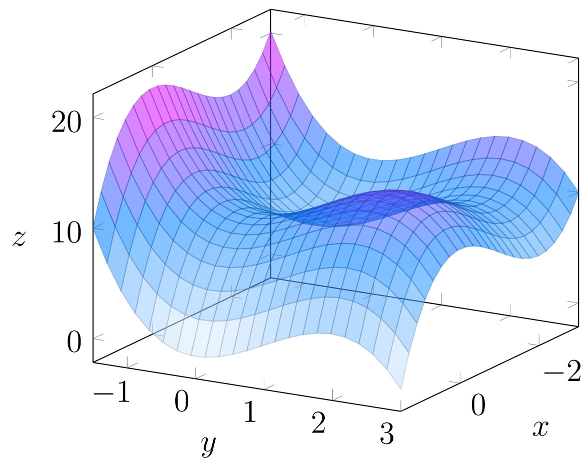
At no point does either f_x or f_y fail to exist, so we search for any point (x,y) for which $f_x(x,y) = f_y(x,y) = 0$. This yields the system

$$\begin{cases} 3x^2 + 6x = 0 \\ 3y^2 - 6y = 0 \end{cases}$$

The system has solutions $(0,0)$, $(0,2)$, $(-2,0)$, and $(-2,2)$. We construct a table:

(x,y)	f_{xx}	f_{yy}	f_{xy}	Φ	Conclusion
$(0,0)$	-6	6	0	-36	Saddle Point
$(0,2)$	-6	-6	0	36	Local Maximum
$(-2,0)$	6	6	0	36	Local Minimum
$(-2,2)$	6	-6	0	-36	Saddle Point

Below is a graph of a part of the surface containing the points of interest.



4 Integrate with respect to y first:

$$\iint_R x^5 e^{x^3 y} dA = \int_0^{\ln 2}\int_0^1 x^5 e^{x^3 y} dy dx = \int_0^{\ln 2} x^2(e^{x^3}-1)dx$$
$$= \int_0^{\ln 2} x^2 e^{x^3} dx - \int_0^{\ln 2} x^2 dx = \frac{e^{\ln^3 2}-1}{3} - \frac{\ln^3 2}{3}$$
$$= \frac{e^{\ln^3 2} - 1 - \ln^3 2}{3}.$$

5 We have

$$\iint_R y^2\, dA = \int_{-1}^{1}\int_{-x-1}^{2x+2} y^2\, dy dx = \int_{-1}^{1} 3(x+1)^3\, dx = \frac{3}{4}(x+1)^4\Big|_{-1}^{1} = 12.$$

6 Area is

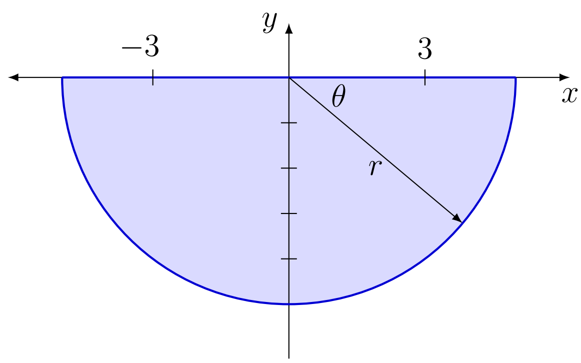
$$\iint_R dA = \int_0^{\ln 2}\int_0^{e^x} dy dx = \int_0^{\ln 2} e^x\, dx = e^{\ln 2} - e^0 = 1.$$

7 The sketch of R in the xy-plane is below. The region

$$S = \{(r,\theta) : 0 \le r \le 5 \text{ and } \pi \le \theta \le 2\pi\}$$

in the $r\theta$-plane is such that $T_{\text{pol}}(S) = R$, and therefore

$$\begin{aligned}\iint_R 2xy\,dA &= \iint_S 2(r\cos\theta)(r\sin\theta)r\,dA = \int_\pi^{2\pi}\int_0^5 2(r\cos\theta)(r\sin\theta)r\,drd\theta \\ &= \int_\pi^{2\pi}\int_0^5 2r^3\cos\theta\sin\theta\,drd\theta = \int_\pi^{2\pi}\cos\theta\sin\theta\left[\frac{1}{2}r^4\right]_0^5 d\theta \\ &= \frac{625}{2}\int_\pi^{2\pi}\cos\theta\sin\theta\,d\theta = \frac{625}{4}\int_\pi^{2\pi}\sin(2\theta)\,d\theta = 0.\end{aligned}$$



8 The height function is

$$h(x) = (27 - x^2 - 2y^2) - (2x^2 + y^2) = 27 - 3x^2 - 3y^2,$$

while the region of integration R will be the region in the xy-plane enclosed by the curve that is the projection onto $z = 0$ of the curve of intersection of the paraboloids. This curve is given by

$$2x^2 + y^2 = 27 - x^2 - 2y^2,$$

or $x^2 + y^2 = 9$, which is a circle with center $(0,0)$ and radius 3, and so in polar coordinates

$$R = \{(r,\theta) : 0 \le \theta \le 2\pi,\;\; 0 \le r \le 3\}.$$

The volume of the solid is

$$\begin{aligned}\mathcal{V} = \iint_R h &= \int_0^{2\pi}\int_0^3 (27 - 3r^2\cos^2\theta - 3r^2\sin^2\theta)r\,drd\theta \\ &= \int_0^{2\pi}\int_0^3 (27r - 3r^3)\,drd\theta = \frac{243}{2}\pi.\end{aligned}$$