Math 242 Exam #2 Key (Fall 2018)

1 For $p_0 = (1,1,0)$, $q_0 = (-2,8,4)$, $r_0 = (1,2,3)$, we have $\overrightarrow{p_0q_0} = \langle -3,7,4\rangle$ and $\overrightarrow{p_0r_0} = \langle 0,1,3\rangle$. Now,

$$\begin{aligned}\mathbf{n} = \overrightarrow{p_0q_0} \times \overrightarrow{p_0r_0} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 7 & 4 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 7 & 4 \\ 1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 4 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 7 \\ 0 & 1 \end{vmatrix}\mathbf{k} \\ &= 17\mathbf{i} + 9\mathbf{j} - 3\mathbf{k} = \langle 17, 9, -3\rangle,\end{aligned}$$

so if $p = (x,y,z)$, then the equation of the plane is given by

$$\mathbf{n} \cdot \overrightarrow{p_0p} = \langle 17,9,-3\rangle \cdot \langle x-1, y-1, z\rangle = 0,$$

or $17x + 9y - 3z = 26$.

2 We have planes $P : x + 2y - 3z = 1$ and $Q : x + y + z = 2$. Now, the intersection of P and the plane $z = 0$ is the set of points on the line $\ell_0 : x + 2y = 1$, and the intersection of Q and $z = 0$ is the line $\ell_0' : x + y = 2$. So the point that is an element of $\ell_0 \cap \ell_0'$ must be a point in $P \cap Q$. We find this point by finding the solution to the system

$$\begin{cases} x + 2y = 1 \\ x + y = 2 \end{cases}$$

which is $(3,-1)$. Thus $(3,-1,0) \in P \cap Q$ (since we're on the plane $z = 0$).

Next, the intersection of P and the plane $z = 1$ is the line $\ell_1 : x + 2y = 4$, and the intersection of Q and $z = 1$ is the line $\ell_1' : x + y = 1$. Again, a point in $\ell_1 \cap \ell_1'$ is a point in $P \cap Q$. The system

$$\begin{cases} x + 2y = 4 \\ x + y = 1 \end{cases}$$

has solution $(-2,3)$, and thus $(-2,3,1) \in P \cap Q$ (recall we're now on the plane where z is 1).

So the line of intersection for P and Q contains points $r_0(3,-1,0)$ and $r_1(-2,3,1)$. Let $\mathbf{v} = \overrightarrow{r_0r_1} = \langle -5,4,1\rangle$. An equation for the line is thus

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 3,-1,0\rangle + t\langle -5,4,1\rangle.$$

3 Domain of F is

$$\{(x,y) : x \geq 0 \text{ and } 1 - x^2 - y^2 \geq 0\} = \{(x,y) : x \geq 0 \text{ and } x^2 + y^2 \leq 1\},$$

which is the right half of the closed disc of radius 1 centered at $(0,0)$. Since F is a combination of radical functions, this is where F is continuous.

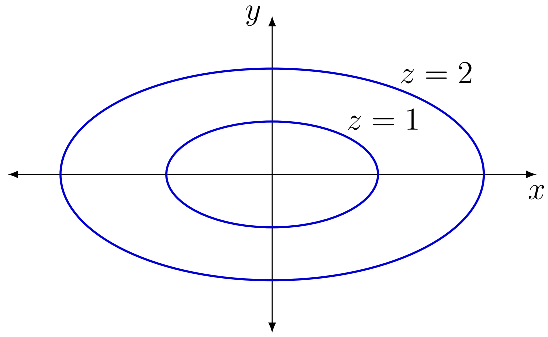
4 The level curve $z = 1$ has equation $1 = \sqrt{x^2 + 4y^2}$, which implies

$$x^2 + \frac{y^2}{1/4} = 1,$$

an ellipse. The level curve $z = 2$ has equation $2 = \sqrt{x^2 + 4y^2}$, which implies

$$\frac{x^2}{4} + y^2 = 1,$$

also an ellipse. Graph is below.



5 The limit becomes

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \cdot \frac{\sqrt{x^2 + y^2 + 1} + 1}{\sqrt{x^2 + y^2 + 1} + 1} \right) = \lim_{(x,y)\to(0,0)} \frac{(x^2 + y^2)(\sqrt{x^2 + y^2 + 1} + 1)}{x^2 + y^2}$$
$$= \lim_{(x,y)\to(0,0)} (\sqrt{x^2 + y^2 + 1} + 1) = \sqrt{0^2 + 0^2 + 1} + 1 = 2.$$

6 First approach $(0, 0)$ on the path $x = y$, so limit becomes

$$\lim_{x\to 0} \frac{x^3 \cos x}{x^2 + x^4} = \lim_{x\to 0} \frac{x \cos x}{1 + x^2} = \frac{0 \cos 0}{1 + 0^2} = 0.$$

Now approach $(0, 0)$ on the path $x = y^2$, so the limit becomes

$$\lim_{y\to 0} \frac{y^4 \cos y}{2y^4} = \lim_{y\to 0} \frac{\cos y}{2} = \frac{\cos 0}{2} = \frac{1}{2}.$$

The limits don't agree, so the original limit cannot exist by the Two-Path Test.

7a We have

$$g_x(x, y) = \ln(x^2 + y^2) + \frac{2x^2}{x^2 + y^2} \quad \text{and} \quad g_y(x, y) = \frac{2xy}{x^2 + y^2}.$$

7b We have

$$h_z(x, y, z) = -3 \sin(x + 2y + 3z) \quad \text{and} \quad h_{zy}(x, y, z) = -2 \cos(x + 2y + 3z).$$

8a Along the path $y = x$ the limit becomes

$$\lim_{(x,x)\to(0,0)} -\frac{x^2}{x^2+x^2} = \lim_{(x,x)\to(0,0)} -\frac{1}{2} = -\frac{1}{2},$$

which implies that

$$\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 0$$

and therefore f is not continuous at $(0,0)$.

8b By an established theorem, since f is not continuous at $(0,0)$ it cannot be differentiable at $(0,0)$.

8c By definition we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0}(0) = 0.$$

Thus, even though f is not differentiable at $(0,0)$, it can have partial derivatives at $(0,0)$.

9 Here $w(t) = f(x,y)$ with $f(x,y) = \cos(2x)\sin(3y)$, $x = x(t) = t/2$ and $y = y(t) = t^4$. By Chain Rule 1 in the notes,

$$\begin{aligned} w'(t) &= f_x(x,y)x'(t) + f_y(x,y)y'(t) = -\sin(2x)\sin(3y) + 12t^3\cos(2x)\cos(3y) \\ &= -\sin(t)\sin(3t^4) + 12t^3\cos(t)\cos(3t^4). \end{aligned}$$

10a $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle -9x^2, 2\rangle$

10b Direction of steepest ascent is

$$\frac{\nabla f(1,2)}{|\nabla f(1,2)|} = \frac{\langle -9,2\rangle}{\sqrt{(-9)^2+2^2}} = \frac{1}{\sqrt{85}}\langle -9,2\rangle\,,$$

and direction of steepest descent is

$$-\frac{1}{\sqrt{85}}\langle -9,2\rangle\,.$$

10c Let C_0 be given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \geq 0$. Then for any t the tangent vector to C_0 at the point $(x(t), y(t))$, which is $\mathbf{r}'(t)$, must be in the direction of $-\nabla f(x,y) = \langle 9x^2(t), -2\rangle$. Therefore we set

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \langle 9x^2(t), -2\rangle,$$

from which we obtain the differential equations $x' = 9x^2$ and $y' = -2$. The first equation can be solved by the Method of Separation of Variables:

$$\frac{dx}{dt} = 9x^2 \;\Rightarrow\; \frac{dx}{9x^2} = dt \;\Rightarrow\; \int \frac{1}{9x^2}\,dx = \int dt \;\Rightarrow\; -\frac{1}{9x} = t + K \;\Rightarrow\; x(t) = -\frac{1}{9t+K},$$

with arbitrary constant K. The equation $y' = -2$ easily gives $y(t) = -2t + K'$ for arbitrary constant K'. Since C is given to start at $(1, 2, 3)$, we must have C_0 start at $(1, 2)$; that is, $\mathbf{r}(0) = \langle x(0), y(0)\rangle = \langle 1, 2\rangle$. From $-1/(9 \cdot 0 + K) = x(0) = 1$ we obtain $K = -1$, and from $-2(0) + K' = y(0) = 2$ we obtain $K' = 2$. Therefore an equation for C_0 is

$$\mathbf{r}(t) = \left\langle \frac{1}{1-9t},\ 2-2t \right\rangle, \quad t \geq 0.$$

11 First get the unit vector in the direction of $\langle 1, \sqrt{3}\rangle$:

$$\mathbf{u} = \frac{\langle 1, \sqrt{3}\rangle}{2} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle.$$

Now,

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u} = \langle e^x \sin y, e^x \cos y\rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \frac{e^x \sin y}{2} + \frac{\sqrt{3}e^x \cos y}{2},$$

and so

$$D_{\mathbf{u}}f(0, \pi/4) = \frac{e^0 \sin(\pi/4)}{2} + \frac{\sqrt{3}e^0 \cos(\pi/4)}{2} = \frac{1/\sqrt{2}}{2} + \frac{\sqrt{3} \cdot 1/\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}.$$