Math 242 Exam #3 Key (Fall 2017)

1 The surface Σ is given by $F(x,y,z)=0$ for $F(x,y,z)=xy^2+3x-z^2-4$. We have

$$\nabla F(x,y,z)=\left\langle y^2+3, 2xy, -2z\right\rangle,$$

and the equation of the tangent plane at $(2,1,-2)$ is given by

$$0=\nabla F(2,1,-2)\cdot\langle x-2, y-1, z+2\rangle=\langle 4,4,4\rangle\cdot\langle x-2, y-1, z+2\rangle,$$

or

$$x+y+z=1.$$

2 These will be points $(x,y,z)\in S$ such that $\nabla F(x,y,z)=\langle 0,0,c\rangle$ for some $c\neq 0$. Here

$$F(x,y,z)=3x^2+2y^2-3x+4y-z-5,$$

so $F_x(x,y,z)=6x-3$, $F_y(x,y,z)=4y+4$, and $F_z(x,y,z)=-1$, and hence

$$\nabla F(x,y,z)=\langle 6x-3, 4y+4, -1\rangle.$$

Now,

$$\nabla F(x,y,z)=\langle 0,0,c\rangle \quad\Rightarrow\quad \langle 6x-3, 4y+4, -1\rangle=\langle 0,0,c\rangle,$$

which requires $x=1/2$ and $y=-1/2$ (also we have $c=-1\neq 0$ as required). On the other hand $(1/2,-1/2,z)\in S$ implies $z=-29/4$. Thus the point $(1/2,-1/2,-29/4)$ on S has a horizontal tangent plane.

3 Note that $f_x(x,y)=-e^{-x}\sin y$ and $f_y(x,y)=e^{-x}\cos y$, and so it's never the case that $f_x(x,y)=f_y(x,y)=0$. Thus there are no critical points, and we conclude that there can be no extrema or saddle points.

4 We have

$$\begin{aligned} f_x(x,y)&=2x+y, & f_y(x,y)&=x,\\ f_{xx}(x,y)&=2, & f_{yy}(x,y)&=0,\\ f_{xy}(x,y)&=1. \end{aligned}$$

At no point does f_x or f_y fail to exist, so we search for any (x,y) for which $f_x(x,y)=f_y(x,y)=0$. This yields the system

$$\begin{cases} 2x+y=0\\ x\quad\;\;=0 \end{cases}$$

The only solution is $(0,0)$, which is our only critical point. Now,

$$f_{xx}(0,0)f_{yy}(0,0)-f_{xy}^2(0,0)=-1<0,$$

so f has a saddle point at $(0,0)$, and hence there are no extrema of any kind in the *open* rectangle $\{(x,y):|x|<2,\ |y|<1\}=(-2,2)\times(-1,1)$.

Next we look at each side of the rectangle. Left side: define $g_1(y)=f(-2,y)=4-2y$ for $y\in[-1,1]$. Minimum is at $g_1(1)=2$, maximum is at $g_1(-1)=6$.

Right side: define $g_2(y) = f(2, y) = 4 + 2y$ for $y \in [-1, 1]$. Minimum is at $g_2(-1) = 2$, maximum is at $g_2(1) = 6$. At this juncture the value of $f(x, y)$ has been determined at all four corners of R.

Bottom side: define $g_3(x) = f(x, -1) = x^2 - x$ for $x \in [-2, 2]$. Minimum is at $g_3(1/2) = -3/4$. Endpoint values have already been evaluated.

Top side: define $g_4(x) = f(x, 1) = x^2 + x$ for $x \in [-2, 2]$. Minimum is at $g_4(-1/2) = -1/4$. Endpoint values have already been evaluated.

Absolute maximum of $f(x, y)$ on R is thus $f(-2, \pm 1) = 6$, and absolute minimum is $f(1/2, -1) = -3/4$.

5 Set $g(x, y, z) = x^2 + y^2 + z^2 - 4$, so the constraint is $g(x, y, z) = 0$. Find all $(x, y, z) \in \mathbb{R}^3$ for which there can be found some $\lambda \in \mathbb{R}$ such that the system

$$\begin{cases} f_x(x, y, z) = \lambda g_x(x, y, z) \\ f_y(x, y, z) = \lambda g_y(x, y, z) \\ f_z(x, y, z) = \lambda g_z(x, y, z) \\ \; g(x, y, z) = 0 \end{cases}$$

has a solution. Explicitly the system is

$$\begin{cases} \;\;\, 1 = 2\lambda x \\ \;\;\, 3 = 2\lambda y \\ -1 = 2\lambda z \\ \;\;\, 4 = \;\; x^2 + y^2 + z^2 \end{cases} \tag{1}$$

First equation gives $\lambda = 1/2x$. Then 2nd and 3rd equations give $y = 3x$ and $z = -x$. From this the 4th equation gives $x = \pm 2/\sqrt{11}$. This leaves us with points $(\pm 2/\sqrt{11}, \pm 6/\sqrt{11}, \mp 2/\sqrt{11})$. Evaluate f at these points to get

$$f\big(2/\sqrt{11}, 6/\sqrt{11}, -2/\sqrt{11}\big) = 22/\sqrt{11},$$

which is the maximum, and

$$f\big(-2/\sqrt{11}, -6/\sqrt{11}, 2/\sqrt{11}\big) = -22/\sqrt{11},$$

which is the minimum.

6 By Fubini's Theorem integral becomes

$$\int_0^{\pi/3} \int_0^1 x \sec^2(xy)\, dy\, dx = \ln 2.$$

7 Integral is

$$\iint_R f = \int_0^1 \int_x^{-x+2} f(x, y)\, dy\, dx$$

8 We have

$$\iint_R 3xy\,dA = \int_0^2 \int_{2-y}^{4-y^2} 3xy\,dx\,dy = 14.$$

9 Integral becomes

$$\int_0^{\pi} \int_1^2 \frac{r}{1+r^2}\,dr\,d\theta = \frac{\pi}{2} \ln \frac{5}{2}.$$