Math 242 Exam #2 Key (Fall 2017)

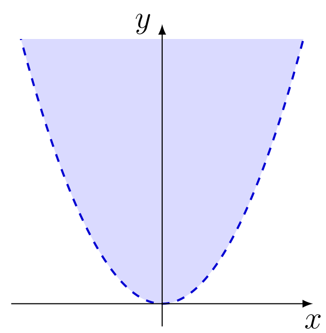
1 From the first equation: $x = z - 2y + 1$. Put this into the second equation to get $y = 2z - 1$. Now put $y = 2z - 1$ into $x = z - 2y + 1$ to get $x = -3z + 3$. Thus the set of points lying at the intersection of the two planes is

$$\{(x, y, z) : x = -3z + 3 \text{ and } y = 2z - 1\} = \{(-3t + 3, 2t - 1, t) : t \in \mathbb{R}\}.$$

Equation for the line:

$$\mathbf{r}(t) = \langle -3t + 3, 2t - 1, t\rangle, \quad t \in \mathbb{R}.$$

2 Domain is $\{(x, y) : x^2/2 < y\}$, shown below.



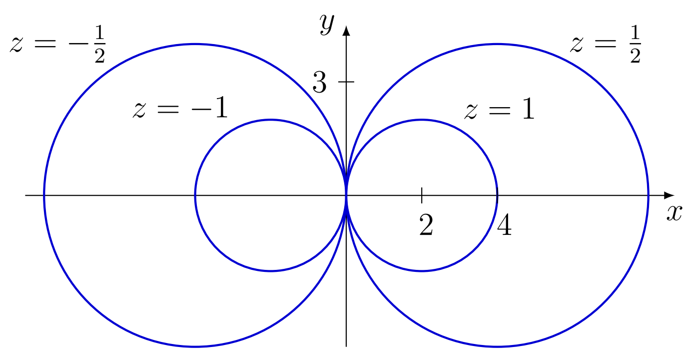
3 Level curve $F(x, y) = \pm 1$ has equation

$$\frac{4x}{x^2 + y^2} = \pm 1 \quad \Rightarrow \quad x^2 + y^2 = \pm 4x \quad \Rightarrow \quad (x \pm 2)^2 + y^2 = 4,$$

a circle with center $(\pm 2, 0)$ and radius 2. Level curve $F(x, y) = \pm\frac{1}{2}$ has equation

$$\frac{4x}{x^2 + y^2} = \pm\frac{1}{2} \quad \Rightarrow \quad x^2 + y^2 = \pm 8x \quad \Rightarrow \quad (x \pm 4)^2 + y^2 = 16,$$

a circle with center $(\pm 4, 0)$ and radius 4.



4 We have

$$\lim_{(x,y)\to(1,1)} \frac{x^2 + xy - 2y^2}{2x^2 - xy - y^2} = \lim_{(x,y)\to(1,1)} \frac{(x - y)(x + 2y)}{(2x + y)(x - y)} = \lim_{(x,y)\to(1,1)} \frac{x + 2y}{2x + y} = \frac{1 + 2(1)}{2(1) + 1} = 1.$$

5 Along the path $y = x$ the limit becomes

$$\lim_{x\to 0} \frac{x^3}{x^2 + x^4} = \lim_{x\to 0} \frac{x}{x^2+1} = 0.$$

Along the path $y = \sqrt{x}$ the limit becomes

$$\lim_{x\to 0^+} \frac{x(\sqrt{x})^2}{x^2 + (\sqrt{x})^4} = \lim_{x\to 0^+} \frac{x^2}{x^2 + x^2} = \lim_{x\to 0^+} \frac{1}{2} = \frac{1}{2}.$$

The limits on the chosen paths are not equal, therefore the limit does not exist by the Two-Path Test.

6 $z_x = e^x \sin y$, $z_y = e^x \cos y$, $z_{xx} = e^x \sin y$, $z_{yy} = -e^x \sin y$, and so $z_{xx} + z_{yy} = 0$.

7a We have $\psi(h, 0) = 0$ and $\psi(0, h) = 0$ for any $h \neq 0$. Now,

$$\psi_x(0,0) = \lim_{h\to 0} \frac{\psi(h,0) - \psi(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

and

$$\psi_y(0,0) = \lim_{h\to 0} \frac{\psi(0,h) - \psi(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

7b Along the path $y = x$ we have

$$\lim_{(x,y)\to(0,0)} \psi(x,y) = \lim_{x\to 0} \frac{8x^3}{x^3 + 2x^3} = \frac{8}{3},$$

so right away we see that

$$\lim_{(x,y)\to(0,0)} \psi(x,y) \neq 0 = \psi(0,0),$$

and therefore ψ is not continuous at $(0, 0)$.

7c Since ψ is not continuous at $(0, 0)$, it cannot be differentiable at $(0, 0)$.

8a From $\nabla f(x,y) = \langle 2x + 4y, 4x - 3y^2\rangle$ we get $\nabla f(-2, 3) = \langle 8, -35\rangle$.

8b Steepest ascent and descent:

$$\frac{\nabla f(-2,3)}{\|\nabla f(-2,3)\|} = \frac{1}{\sqrt{1289}}\langle 8, -35\rangle \quad \text{and} \quad -\frac{\nabla f(-2,3)}{\|\nabla f(-2,3)\|} = -\frac{1}{\sqrt{1289}}\langle 8, -35\rangle,$$

respectively. No change: we need a vector orthogonal to $\langle 8, -35\rangle$, such as $\langle 35, 8\rangle$, then normalize to get

$$\frac{1}{\sqrt{1289}}\langle 35, 8\rangle.$$

9 The path is a curve C given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \geq 0$, where $\mathbf{r}(0) = \langle 10, 10\rangle$. For C to be the path of steepest ascent (i.e. greatest temperature increase), for any $t \geq 0$ the tangent vector to C at $\mathbf{r}(t)$, which is $\mathbf{r}'(t)$, must be in the direction of

$$\nabla T(\mathbf{r}(t)) = \nabla T(x(t), y(t)) = \big\langle T_x(x(t), y(t)), T_y(x(t), y(t))\big\rangle = \langle -4x(t), -2y(t)\rangle.$$

So we set

$$\mathbf{r}'(t) = \langle -4x(t), -2y(t)\rangle,$$

which gives the differential equations $x'(t) = -4x(t)$ and $y'(t) = -2y(t)$. We treat the first equation using Separation of Variables:

$$\frac{dx}{dt} = -4x \quad\Rightarrow\quad \int \frac{1}{x}\,dx = -\int 4\,dt \quad\Rightarrow\quad \ln|x| = -4t + c,$$

c an arbitrary constant. Since $x > 0$ at the initial point $(10, 10)$, we have $|x| = x$, and thus $\ln x = -4t + c$. Now,

$$\ln x = -4t + c \quad\Rightarrow\quad x(t) = e^{-4t+c} = Ke^{-4t},$$

where $K = e^c$. From the initial condition $x(0) = 10$ we find that $K = 10$, and hence $x(t) = 10e^{-4t}$. A nearly identical routine shows that $y' = -2y$ has solution $y(t) = 10e^{-2t}$. Now we have

$$\mathbf{r}(t) = \langle 10e^{-4t}, 10e^{-2t}\rangle, \quad t \in [0, \infty). \tag{1}$$

This is a good answer.

There's the option of going farther. We see in (1) that $x/10 = e^{-4t}$ and $y/10 = e^{-2t}$, so that $(y/10)^2 = x/10$, and hence $y^2 = 10x$. This eliminates the parameter t, but the whole graph of $y^2 = 10x$ is not exactly C. In fact, x starts at 10, and then decreases in value as t increases since $x'(t) = -40e^{-4t} < 0$. Therefore C is precisely given by.

$$y^2 = 10x, \quad x \in (-\infty, 10].$$