Math 242 Exam #1 Key (Fall 2017)

1a Plane: $\mathbf{p} = \langle 0, 300, 0\rangle$, downdraft: $\mathbf{d} = \langle 0, 0, -32\rangle$. The crosswind $\mathbf{c}$ is in the direction $\langle 1, 1, 0\rangle$ with magnitude 55, so $\mathbf{c} = \frac{55}{\sqrt{2}}\langle 1, 1, 0\rangle$. Resultant velocity $\mathbf{v}$ of the plane is

$$\mathbf{v} = \mathbf{p} + \mathbf{c} + \mathbf{d} = \left\langle \frac{55}{\sqrt{2}},\ 300 + \frac{55}{\sqrt{2}},\ -32 \right\rangle.$$

Speed is $\|\mathbf{v}\| \approx 342.6 \approx 343$ km/h.

1b A vector parallel to the horizontal that is also parallel to $\mathbf{v}$ is

$$\mathbf{u} = \sqrt{2}\mathbf{v} = \left\langle \frac{55}{\sqrt{2}},\ 300 + \frac{55}{\sqrt{2}},\ 0 \right\rangle.$$

Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \approx \frac{116,359.5}{(341.1)(342.6)} \approx 0.9957,$$

and so $\theta \approx \cos^{-1}(0.9957) \approx 5.3°$.

2 Find x and y such that $x\mathbf{u} + y\mathbf{v} = \langle a, b\rangle$. This gives rise to the system

$$\begin{cases} x - 2y = a \\ x + \ \ y = b \end{cases}$$

Adding twice the second equation to the first equation yields $3x = a + 2b$, so $x = (a + 2b)/3$, and then since $y = b - x$ we obtain

$$x = \frac{a + 2b}{3} \quad \text{and} \quad y = \frac{b - a}{3}.$$

Thus

$$\langle a, b\rangle = \left(\frac{a + 2b}{3}\right)\mathbf{u} + \left(\frac{b - a}{3}\right)\mathbf{v}.$$

3 Center of sphere is at $(-2, 2, 5)$. Radius is

$$\frac{1}{2}\|\mathbf{p} - \mathbf{q}\| = \frac{1}{2}\sqrt{4^2 + 0^2 + 4^2} = \frac{\sqrt{32}}{2} = 2\sqrt{2}.$$

Equation for the sphere:

$$(x + 2)^2 + (y - 2)^2 + (z - 5)^2 = 8.$$

4 First find a parametrization for the line containing the known points:

$$\mathbf{r}(t) = \langle 1, 2, 3\rangle + t\big(\langle 4, 7, 1\rangle - \langle 1, 2, 3\rangle\big) = \langle 1, 2, 3\rangle + t\langle 3, 5, -2\rangle = \langle 1 + 3t, 2 + 5t, 3 - 2t\rangle.$$

Now, x and y must be such that $\mathbf{r}(t) = \langle x, y, 9\rangle$ for some t. That is,

$$\begin{cases} 1 + 3t = x \\ 2 + 5t = y \\ 3 - 2t = 9 \end{cases}$$

The last equation gives $t = -3$, and so we find that $x = -8$ and $y = -13$.

5a $\|\mathbf{u}\| = \sqrt{2^2+1^2+9^2} = \sqrt{86}$ and $\|\mathbf{v}\| = \sqrt{2^2+5^2+3^2} = \sqrt{38}$. Now,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{16}{\sqrt{5}\sqrt{89}} \quad\Rightarrow\quad \theta = \cos^{-1} 0.7585) \approx 40.67°.$$

5b We have

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{16}{89}\langle 4,-8,3\rangle.$$

6 The area of the parallelogram is

$$\|\mathbf{u}\times\mathbf{v}\| = \left\| \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 0 \\ 1 & 1 \end{vmatrix}\mathbf{k} \right\| = \sqrt{(-2)^2+5^2+(-3)^2} = \sqrt{38}.$$

7 The direction vector $\mathbf{v}$ of the line is perpendicular to $\mathbf{k} = \langle 0,0,1\rangle$, so is parallel to $\mathbf{u}\times\mathbf{k}$. Thus we can let

$$\mathbf{v} = \mathbf{u}\times\mathbf{k} = \begin{vmatrix} 3 & -5 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & -5 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & 3 \\ 0 & 0 \end{vmatrix}\mathbf{k} = \langle 3,-4,0\rangle,$$

and a parametrization is

$$\mathbf{r}(t) = \langle 0,2,1\rangle + t\langle 3,-4,0\rangle = \langle 3t, -4t+2, 1\rangle, \quad t\in\mathbb{R}.$$

8 The direction vectors of the lines are $\langle 2,3,-1\rangle$ and $\langle -3,-4,-2\rangle$, which are not parallel vectors, and so the lines are not parallel. The lines intersect if and only if there is some s and t such that $\mathbf{r}(t) = \mathbf{R}(s)$. This gives us the system

$$\begin{cases} 5+2t = 13-3s \\ 3+3t = 13-4s \\ 1-\ \ t = \ \ 4-2s \end{cases}$$

The 3rd equation gives $t = 2s-3$. Putting this into the 1st equation results in $s = 2$, and hence $t = 1$. However, putting $(s,t) = (2,1)$ into the 2nd equation results in $6 = 5$, so there is no solution to the system. The lines do not intersect, and therefore are skew.

9 We have

$$\mathbf{r}(t) = \left\langle \frac{1}{2}e^{2t} + c_1,\ t + 2e^{-t} + c_2,\ t - 2e^t + c_3 \right\rangle,$$

so $\mathbf{r}(0) = \langle \frac{1}{2}+c_1, 2+c_2, -2+c_3\rangle$. Since $\mathbf{r}(0) = \langle 1,1,1\rangle$, it follows that $c_1 = \frac{1}{2}$, $c_2 = -1$, $c_3 = 3$. Therefore

$$\mathbf{r}(t) = \left\langle \frac{1}{2}e^{2t} + \frac{1}{2},\ t + 2e^{-t} - 1,\ t - 2e^t + 3 \right\rangle.$$

10 Since $\mathbf{r}'(t) = \langle -10t\sin t^2, 10t\cos t^2, 24t\rangle$, speed is

$$\|\mathbf{r}'(t)\| = \sqrt{100t^2 + 576t^2} = 26t.$$

Length of trajectory:

$$\int_0^2 \|\mathbf{r}'(t)\|\,dt = \left[13t^2\right]_0^2 = 52.$$

11 First, $\mathbf{r}'(t) = \langle 1, 2t, t\rangle$, so $\|\mathbf{r}'(t)\| = \sqrt{1+5t^2}$. Then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1+5t^2}}\langle 1, 2t, 2\rangle$$

implies

$$\mathbf{T}'(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{(1+5t^2)^{3/2}}\langle -5t, 2, 1\rangle,$$

and thus

$$\|\mathbf{T}'(t)\| = \frac{\sqrt{5}}{1+5t^2}.$$

Finally, the curvature is

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{5}}{1+5t^2}\cdot\frac{1}{\sqrt{1+5t^2}} = \frac{\sqrt{5}}{(1+5t^2)^{3/2}}.$$

The curvature is maximal when $1+5t^2$ is minimal, which occurs when $t = 0$. Thus $\kappa(0) = \sqrt{5}$ is the maximum curvature.