Math 242 Exam #5 Key (Fall 2016)

1 The vector field $\mathbf{F}(x,y,z) = \nabla(xyz)$ is conservative, with potential function $\varphi(x,y,z) = xyz$. The Fundamental Theorem of Line Integrals gives

$$\int_C \nabla(xyz)\cdot d\mathbf{r} = \varphi(\mathbf{r}(\pi)) - \varphi(\mathbf{r}(0)) = \varphi(-1,0,1) - \varphi(1,0,0) = 0 - 0 = 0.$$

2 Get R be the region enclosed by the square C (which includes the points on C itself). Let I be the given line integral. By Green's Theorem,

$$\begin{aligned} I &= \iint_R \left[\partial_x(x^3+xy) - \partial_y(2y^2 - 2x^2y)\right] dA = \iint_R (5x^2 - 3y)\,dA \\ &= \int_{-1}^{1}\int_{-1}^{1}(5x^2 - 3y)\,dx\,dy = \int_{-1}^{1}\left(\frac{10}{3} - 6y\right)dy = \frac{20}{3}. \end{aligned}$$

3a Setting $\mathbf{F} = \langle f, g, h\rangle$, we have

$$(\operatorname{div}\mathbf{F})(x,y,z) = (\nabla\cdot\mathbf{F})(x,y,z) = \partial_x f(x,y,z) + \partial_y g(x,y,z) + \partial_z h(x,y,z) = 2y + 12xz^2.$$

3b Again setting $\mathbf{F} = \langle f, g, h\rangle$,

$$\begin{aligned} (\operatorname{curl}\mathbf{F})(x,y,z) = (\nabla\times\mathbf{F})(x,y,z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix} \\ &= \begin{vmatrix} \partial_y & \partial_z \\ g & h \end{vmatrix}\mathbf{i} - \begin{vmatrix} \partial_x & \partial_z \\ f & h \end{vmatrix}\mathbf{j} + \begin{vmatrix} \partial_x & \partial_y \\ f & g \end{vmatrix}\mathbf{k} \\ &= (\partial_y h - \partial_z g)\mathbf{i} - (\partial_x h - \partial_z f)\mathbf{j} + (\partial_x g - \partial_y f)\mathbf{k} \\ &= \langle h_y - g_z, f_z - h_x, g_x - f_y\rangle = \langle 0-0, 4z^3 - 4z^3, 2x - 2x\rangle \\ &= \langle 0,0,0\rangle = \mathbf{0}. \end{aligned}$$

4a For each $z \in [0,8]$ we have $x^2 + y^2 = z/2$, a circle of radius $\sqrt{z/2}$ with center on the z-axis. Such a circle we may parametrize by

$$\left\langle \sqrt{\frac{z}{2}}\cos t, \sqrt{\frac{z}{2}}\sin t \right\rangle, \quad t \in [0, 2\pi].$$

Let $z = v$ and $t = u$. Then

$$\mathbf{r}(u,v) = \left\langle \sqrt{\frac{v}{2}}\cos u, \sqrt{\frac{v}{2}}\sin u, v \right\rangle, \quad (u,v) \in [0,2\pi]\times[0,8]$$

is a parametrization of the surface Σ.

Alternatively we may replace v with $2v^2$ to obtain the parametrization

$$\mathbf{r}(u,v) = \langle v\cos u, v\sin u, 2v^2\rangle, \quad (u,v) \in [0,2\pi]\times[0,2] \tag{1}$$

(note the correspondingly altered domain).

4b Using the parametrization (1) above, so that

$$\mathbf{r}_u(u,v) = \langle -v\sin u, v\cos u, 0\rangle \quad \text{and} \quad \mathbf{r}_v(u,v) = \langle \cos u, \sin u, 4v\rangle,$$

we find the area of Σ to be

$$\begin{aligned}\mathcal{A} &= \iint_\Sigma dS = \iint_R \|(\mathbf{r}_u \times \mathbf{r}_v)(u,v)\|\, dA = \iint_R v\sqrt{16v^2+1}\, dA \\ &= \int_0^2 \int_0^{2\pi} v\sqrt{16v^2+1}\, du\, dv = \int_0^2 2\pi v\sqrt{16v^2+1}\, dv = \frac{\pi}{24}\big(65\sqrt{65}-1\big).\end{aligned}$$

5 The boundary of Σ, denoted by $\partial\Sigma$, is the set of points (x,y,z) for which $x^2+y^2+z^2 = 25$ and $x = 3$. This is the circle $y^2+z^2 = 16$ at $x = 3$, and a parametrization for $\partial\Sigma$ that is consistent with the orientation of Σ is

$$\mathbf{r}(t) = \langle 3, 4\cos t, 4\sin t\rangle, \quad t \in [0, 2\pi].$$

By Stokes' Theorem,

$$\begin{aligned}\iint_\Sigma (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^{2\pi} \langle 8\cos t, -4\sin t, 3 - 4\cos t - 4\sin t\rangle \cdot \langle 0, -4\sin t, 4\cos t\rangle\, dt \\ &= \int_0^{2\pi} (12\cos t + 16\sin^2 t - 16\cos^2 t - 16\sin t\cos t)\, dt \\ &= \int_0^{2\pi} (12\cos t - 16\cos 2t - 16\sin t\cos t)\, dt = 0.\end{aligned}$$

6 Let D_1 be the region in the first octant between the planes $z = 4 - x - y$ and $z = 0$, and let D_2 be the region in the first octant between the planes $z = 2 - x - y$ and $z = 0$. Then $D = D_1 - D_2$, and since

$$\iiint_{D_1} \nabla \cdot \mathbf{F}\, dV = \int_0^4 \int_0^{4-x} \int_0^{4-x-y} (2x - 2y + 2z)\, dz\, dy\, dx = \frac{64}{3}$$

and

$$\iiint_{D_2} \nabla \cdot \mathbf{F}\, dV = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} (2x - 2y + 2z)\, dz\, dy\, dx = \frac{4}{3},$$

the flux across the boundary of D is

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_{D_1} \nabla \cdot \mathbf{F}\, dV - \iiint_{D_2} \nabla \cdot \mathbf{F}\, dV = \frac{64}{3} - \frac{4}{3} = 20.$$