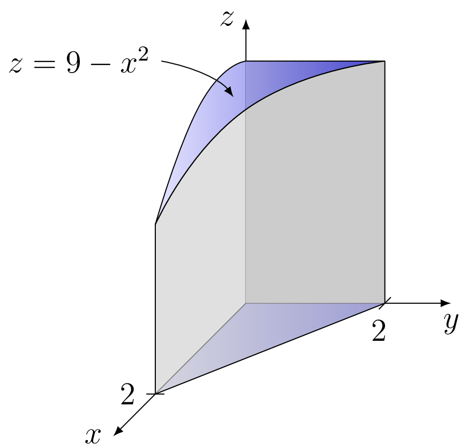
Math 242 Exam #4 Key (Fall 2016)

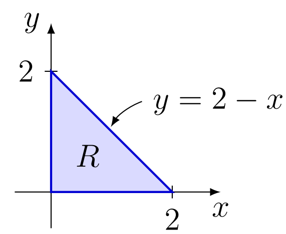
1 For any $(x,y,z) \in D$ we have $0 \le z \le 9 - x^2$. We can evaluate $\iiint_D dV$ in the order $dz\,dy\,dx$ (other orders are possible). See the figure below.

To determine the limits of integration for y and x, project D onto the xy-plane to obtain the region R shown at right in the figure. There it can be seen that if $(x,y) \in R$, then $0 \le y \le 2 - x$ for $0 \le x \le 2$, and so the limits of integration for y will be 0 and $2-x$, and the limits of integration for x will be 0 and 2. We obtain

$$\begin{aligned}\mathcal{V}(D) &= \iiint_D dV = \int_0^2 \int_0^{2-x} \int_0^{9-x^2} dz\,dy\,dx \\ &= \int_0^2 \int_0^{2-x} (9 - x^2)\,dy dx = \int_0^2 \left[9y - x^2 y\right]_0^{2-x} dx \\ &= \int_0^2 \left[9(2-x) - x^2(2-x)\right] dx = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{9}{2}x^2 + 18x\right]_0^2 = \frac{50}{3}.\end{aligned}$$

It can be instructive to try determining the volume of D by integrating in the orders $dz\,dx\,dy$ and $dy\,dz\,dx$.

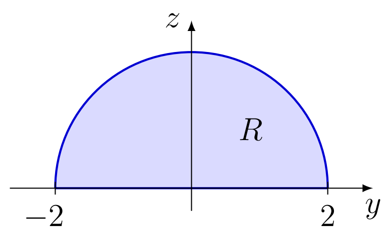




2 On the yz-plane the region of integration is

$$R = \left\{(y,z) : 0 \le z \le \sqrt{4 - y^2},\ -2 \le y \le 2\right\},$$

the top half of a circular disc of radius 2:



This region is also expressible as

$$R = \left\{(y,z) : -\sqrt{4 - z^2} \le y \le \sqrt{4 - z^2},\ 0 \le z \le 2\right\},$$

and so the integral becomes

$$\int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\, dz\, dx.$$

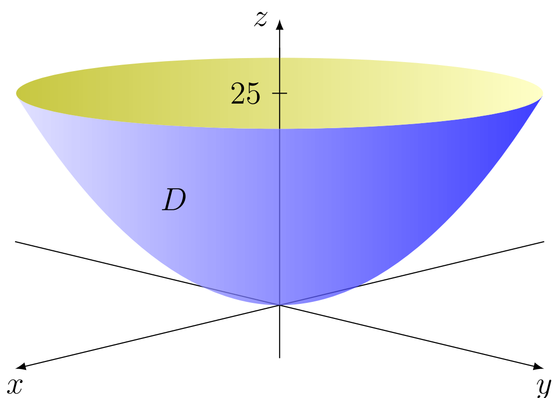
To evaluate the integral let $z = 2\sin\theta$, so that $dz = 2\cos\theta\, d\theta$, and we obtain

$$\begin{aligned}\int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\, dz\, dx &= \int_0^1 \left(\int_0^{\pi/2} 2\sqrt{4-4\sin^2\theta} \cdot 2\cos\theta\, d\theta \right) dx \\ &= \int_0^1 \left(8 \int_0^{\pi/2} \cos^2\theta\, d\theta \right) dx = 8 \int_0^1 \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta\, dx \\ &= 8 \int_0^1 \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4} \right]_0^{\pi/2} dx = 2\pi.\end{aligned}$$

3 The mass m of the cone is

$$\begin{aligned} m &= \int_0^{2\pi} \int_0^6 \int_0^{6-r} (8-z) r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^6 \left(30r - 2r^2 - \tfrac{1}{2}r^3\right) dr\, d\theta \\ &= \int_0^{2\pi} \left[15r^2 - \tfrac{2}{3}r^3 - \tfrac{1}{8}r^4\right]_0^6 d\theta = \int_0^{2\pi} 234\, d\theta = 468\pi.\end{aligned}$$

4a The region D is shown below.



4b The volume V is given by

$$\int_0^{2\pi} \int_0^5 \int_{r^2}^{25} r\, dz dr d\theta.$$

The reasoning is as follows. In cylindrical coordinates the equation of the paraboloid becomes

$$z = x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2,$$

while the equation of the plane remains $z = 25$.

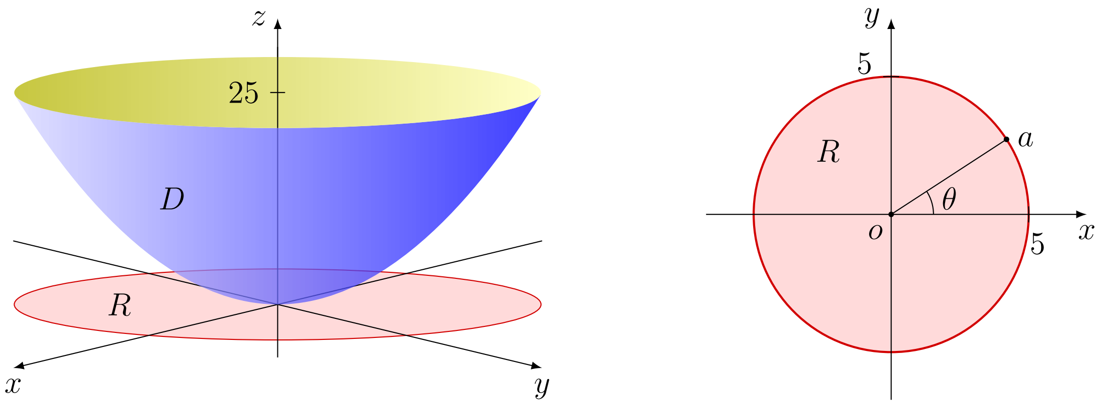


FIGURE 1

The intersection of $z = 25$ and $z = x^2 + y^2$ is the set $\{(x, y, 25) : x^2 + y^2 = 25\}$, which is a curve that projects onto the xy-plane as a circle of radius 5 centered at the origin. Thus the projection of D onto the xy-plane is a region R that is a closed disc with radius 5 centered at the origin, shown at left in Figure 1. A point in R has a θ-coordinate value ranging from $\theta = 0$ to $\theta = 2\pi$; that is, if $(r, \theta) \in R$, then $0 \le \theta \le 2\pi$.

If we fix $\theta \in [0, 2\pi]$, then a point $(r, \theta) \in R$ must lie on the line segment joining $o = (0, 0)$ and $a = (5, \theta)$, shown at right in Figure 1. That is, given $\theta \in [0, 2\pi]$, a point $(r, \theta) \in R$ can have r-coordinate value ranging from $r = 0$ to $r = 5$, so that $0 \le r \le 5$.

Finally, fixing $\theta \in [0, 2\pi]$ and $r \in [0, 5]$, we consider the limits on z in order for (r, θ, z) to be a point in D. We find z must be such that (r, θ, z) is above the paraboloid $z = r^2$ and below $z = 25$, so $r^2 \le z \le 25$.

Thus the set

$$\{(r, \theta, z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 5,\ r^2 \le z \le 25\}$$

is D in cylindrical coordinates.

4c The volume V is

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^5 \int_{r^2}^{25} r \, dz dr d\theta \\ &= \int_0^{2\pi} \int_0^5 (25r - r^3) \, dr d\theta = \int_0^{2\pi} \left[\frac{25}{2} r^2 - \frac{1}{4} r^4 \right]_0^5 d\theta \\ &= \int_0^{2\pi} \frac{625}{4} \, d\theta = \frac{625}{4} \cdot 2\pi = \frac{625}{2} \pi. \end{aligned}$$

5 Let $(x, y) \in C$. A vector parallel to C at the point (x, y) would be $\langle y, -x \rangle$, whereas $\mathbf{F}(x, y) = \langle y, x \rangle$. So $\mathbf{F}(x, y)$ is tangent to C at (x, y) if $\mathbf{F}(x, y) = c\langle y, -x \rangle$ for some $c \neq 0$. That is, $\langle y, x \rangle = c\langle y, -x \rangle$, giving $y = cy$ and $x = -cx$. From $y = cy$ there are two possibilities: $c = 1$ or $y = 0$. If $c = 1$, then $x = -x$ results, and hence $x = 0$. Then, since $x^2 + y^2 = 1$, it follows that $y = \pm 1$, and we obtain two points: $(0, \pm 1)$. If $y = 0$, then $x^2 + y^2 = 1$ implies that $x = \pm 1$, and we obtain another two points: $(\pm 1, 0)$. That is, $\mathbf{F}$ is tangent to C at the four points $(\pm 1, 0)$, $(0, \pm 1)$.

$\mathbf{F}(x, y)$ is normal to C at (x, y) if $\mathbf{F}(x, y) \cdot \langle y, -x \rangle = 0$, which yields $y^2 - x^2 = 0$. Adding this equation to $x^2 + y^2 = 1$ gives $2y^2 = 1$, or $y = \pm 1/\sqrt{2}$. On the other hand $y^2 = x^2$ implies $|x| = |y|$, and so we obtain four points:

$$\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \quad \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right), \quad \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \quad \left(-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right).$$

Note: this problem can also be resolved by working with a parametrization for C, such as the function $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$.

6a A fine parametrization would be

$$\mathbf{r}(t) = \langle 0, -3, 2 \rangle (1 - t) + \langle 1, -7, 4 \rangle t = \langle t, -4t - 3, 2t + 2 \rangle, \quad t \in [0, 1].$$

6b We have $\mathbf{r}'(t) = \langle 1, -4, 2 \rangle$, so that $\|\mathbf{r}'(t)\| = \sqrt{21}$. Now,

$$\begin{aligned}\int_C (xz - y^2)\, ds &= \sqrt{21} \int_0^1 \left[t(2t + 2) - (-4t - 3)^2 \right] dt \\ &= -\sqrt{21} \int_0^1 (14t^2 + 22t + 9)\, dt = -\frac{74\sqrt{21}}{3}.\end{aligned}$$

7 Making the substitution $u = t^2 - 1$ along the way, we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 \mathbf{F}(t^2, t^3) \cdot \langle 2t, 3t^2 \rangle\, dt \\ &= \int_0^1 \langle e^{t^2 - 1}, t^5 \rangle \cdot \langle 2t, 3t^2 \rangle\, dt = \int_0^1 (2te^{t^2 - 1} + 3t^7)\, dt \\ &= \int_0^1 2te^{t^2 - 1}\, dt + \int_0^1 3t^7\, dt = \int_{-1}^0 e^u\, du + \frac{3}{8}\left[t^8\right]_0^1 = \frac{11e - 8}{8e}.\end{aligned}$$

8 Here we have $x(t) = 2\cos t$ and $y(t) = 2\sin t$, so $x'(t) = -2\sin t$ and $y'(t) = 2\cos t$, and then

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{n} &= \int_0^{2\pi} \left[f(\mathbf{r}(t)) y'(t) - g(\mathbf{r}(t)) x'(t) \right] dt \\ &= \int_0^{2\pi} \left[f(2\cos t, 2\sin t)(2\cos t) - g(2\cos t, 2\sin t)(-2\sin t) \right] dt \\ &= \int_0^{2\pi} \left[(2\sin t - 2\cos t)(2\cos t) - (2\cos t)(-2\sin t) \right] dt \\ &= 4\int_0^{2\pi} 2\cos t \sin t\, dt - 4\int_0^{2\pi} \cos^2 t\, dt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \sin(2t)\,dt - 4\int_0^{2\pi} \frac{1+\cos(2t)}{2}\,dt \\
&= 4\left[-\frac{1}{2}\cos(2t)\right]_0^{2\pi} - 2\left[t + \frac{1}{2}\sin(2t)\right]_0^{2\pi} \\
&= 4\cdot 0 - 2\cdot 2\pi = -4\pi.
\end{aligned}
$$

Thus there is a net flux of 4π *into* the region enclosed by C.